Collisions in 3D Fluid Structure interactions problems

Matthieu Hillairet, Takéo Takahashi

COLLISIONS IN 3D FLUID STRUCTURE INTERACTIONS PROBLEMS

MATTHIEU HILLAIRET[†] AND TAKÉO TAKAHASHI [‡§]

Abstract. This paper deals with the system composed by a rigid ball moving into a viscous incompressible fluid, over a fixed horizontal plane. The equations of motion for the fluid are the Navier–Stokes equations and the equations for the motion of the rigid ball are obtained by applying Newton's laws. We show that for any weak solutions of the corresponding system satisfying the energy inequality, the rigid ball never touches the plane. This is the equivalent result to the one obtained in [8] in the 2D setting.

Key words. Fluid structure interactions, Cauchy theory, Qualitative properties, Collisions

AMS subject classifications. 35R35, 76D03, 76D05

1. Introduction. In the last decade, several studies emphasized collisions between rigid bodies in a fluid would lead to great difficulties in the mathematical treatment of fluid-structure interaction models. On the one hand, in [4], E. Feireisl considers a rigid sphere surrounded by a compressible viscous fluid inside a cavity. He constructs a solution to the subsequent system in which the sphere sticks to the ceiling of the cavity without falling down. On the other hand, V.N. Starovoitov proved the fluid-structure interaction system is ill-posed in case of collision [10, 11]. More precisely, he shows there exists too many solutions to the problem when contact occurs. Actually, this corresponds to the lack of collision law in the model under consideration.

Recently, several studies proved lack of collision in fluid structure systems. In [14], E. Zuazua and J.L. Vazquez prove no collision can occur between particles for a 1D toy-model. Then, V.N. Starovoitov obtains a criterion for the velocity-field of solutions [11]. Namely, he proves no collision can occur if the gradient of the velocity-field is sufficiently integrable. Finally, two parallel studies [7, 8] proved a no collision result when there is only one body in a bounded (or partially bounded) two-dimensional cavity. In the first case, the author considers a rigid disk inside a bigger disk. In the second case, the author considers a rigid disk above a ramp. The aim of the present study is to extend these two-dimensional results to three-dimensional comparable configurations i.e., for a rigid sphere above a ramp in $\mathbb{R}^3$.

1.1. Mathematical model. We consider a homogeneous rigid sphere $\mathcal{B}$ with radius 1 and density $\rho_{\mathcal{B}}$. We denote by $\mathbf{G}$ its center (of mass), by $\mathbf{V}$ (resp. $\boldsymbol{\omega}$) its translation (resp. angular) velocity and by m (resp. J) its mass (resp. inertia). Notice $\boldsymbol{\omega}$ is a vector in $\mathbb{R}^3$ and J is a scalar matrix. So, we might identify it with a scalar. The velocity-field of $\mathcal{B}$ reads $\mathbf{V}+\boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G})$ for all $\mathbf{x}\in\mathcal{B}$. The sphere evolves over a ramp $\mathcal{P}$. The remainder of the cavity $\mathbb{R}^3_+$, denoted by $\mathcal{F}$, is filled with an incompressible viscous and Newtonian fluid. It sticks to the boundaries and has constant density $\rho_{\mathcal{F}}=1$, and viscosity μ. Its behavior is described by a velocity/pressure field $(\mathbf{u},p)$. The whole system evolves only through the interactions between solid and fluid without any extern force field.

The evolution of the fluid is prescribed by the incompressible Navier–Stokes equations:

$$\left\{\begin{array}{rcl} \partial_t\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u} &=& \operatorname{div}\mathbb{T}(\mathbf{u},p)\\ \operatorname{div}\mathbf{u} &=& 0\end{array}\right. \qquad \text{in } \mathcal{F}, \tag{1.1}$$

where $\mathbb{T}(\mathbf{u},p)$ is the Newtonian stress tensor:

$$\mathbb{T}(\mathbf{u},p)=2\mu D(\mathbf{u})-p\mathrm{I}_3.$$

[†]Université de Toulouse ; UPS ; IMT ; Laboratoire MIP, Université Paul Sabatier Toulouse 3, 31062 Toulouse Cedex 9 France

[‡]Institut Elie Cartan University of Nancy-I, POB 239, Vandœuvre-lès-Nancy 54506, France

[§]INRIA Nancy

Here $D(\mathbf{u})$ stands for the symmetric part of the gradient of $\mathbf{u}$ while I_3 is the 3x3 identity matrix.

The sphere $\mathcal{B}$ evolves under the action of the fluid only. So, applying the fundamental mechanics principle to $\mathcal{B}$, it yields:

$$\left\{\begin{array}{rcl} -\displaystyle\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma & = & m\dot{\mathbf{V}}, \\ -\displaystyle\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\times(\mathbf{x}-\mathbf{G})\ \mathrm{d}\sigma & = & J\dot{\boldsymbol{\omega}}. \end{array}\right. \tag{1.2}$$

Here $\mathbf{n}$ stands for the normal to $\partial\mathcal{B}$ directed towards $\mathcal{B}$. As J is a scalar matrix the inertial term $J\boldsymbol{\omega}\times\boldsymbol{\omega}$ in the conservation of momentum vanishes.

This system is complemented with boundary conditions:

$$\mathbf{u}_{|\partial\mathcal{B}} = \mathbf{V} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}), \quad \mathbf{u}_{|\mathcal{P}} = 0, \quad \mathbf{u}_{|\infty} = 0 \tag{1.3}$$

and initial conditions:

$$\mathbf{u}(0,\cdot) = \mathbf{u}^0, \quad \mathbf{V}(0) = \mathbf{V}^0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}^0, \quad \mathbf{G}(0) = \mathbf{G}^0. \tag{1.4}$$

The whole system (1.1-1.3) is denoted by (FSIS). We emphasize this system is strongly coupled. On the one hand, the sphere $\mathcal{B}$ fixes the domain $\mathcal{F}$ where the incompressible Navier–Stokes equations (1.1) have to be solved. In particular this domain changes with time. In the following, we denote by $\mathcal{F}(t)$ (resp. $\mathcal{B}(t)$) the domain occupied by the fluid (resp. the solid body) at time t. We shall reserve the notations $\mathcal{B}$ (resp. $\mathcal{F}$) for the sphere (resp. the fluid) as "actors" in the scenarios provided by our solutions to (FSIS). Moreover, the movement of $\mathcal{B}$ fixes the boundary conditions for (1.1) on $\partial\mathcal{B}(t)$. On the other hand, the solution $(\mathbf{u},p)$ prescribes the displacement of $\mathcal{B}$ via the computation of the forces and torques applied to $\mathcal{B}$:

$$-\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ \mathrm{d}\sigma, \quad -\int_{\partial\mathcal{B}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\times(\mathbf{x}-\mathbf{G})\ \mathrm{d}\sigma$$

Our main objective is to prove no collision can occur between $\mathcal{B}$ and $\mathcal{P}$ in finite time in solutions to (FSIS). It reads:

THEOREM 1.1. *Given $T>0$, let $(\mathbf{u},\mathbf{G})$ be a weak solution to (FSIS) over $(0,T)$ with initial data $(\mathbf{u}^0,\mathbf{G}^0)$. Then, there exists a decreasing function $h_{min}\in\mathcal{C}([0,T];\mathbb{R}_+^*)$ depending only on initial data $(\mathbf{u}^0,\mathbf{G}^0)$ such that $h(t) := \mathrm{dist}(\mathcal{B}(t),\mathcal{P})$ satisfies:*

$$h(t) \geqslant h_{min}(t) \quad \forall\, t\in(0,T).$$

This result is expected since computations by M.D.A. Cooley and M.E. O'Neill [1] in the slow motion regime. However, no rigorous mathematical result is yet available in the full nonlinear case. Our proof is based on the choice of a test-function for (FSIS). In section 3, we provide an interpretation for the weak formulation of (FSIS) explaining how the distance can be estimated from below with a suitable test-function. Then, we construct a test-function explicitly. In section 4, the interpretation of the weak formulation is applied to the constructed test-function. Technical details are postponed to Appendices.

As mentioned in Theorem 1.1, our result applies to any weak solution to (FSIS). However the Cauchy theory of weak solutions has been developed only in bounded and in exterior domains (to our knowledge). For the sake of completeness, we extend classical results for the Cauchy theory of (FSIS) to a half-space in next section. So, the no-frontal collision in Theorem 1.1 applies to a non-empty set.

1.2. Notations. In the whole paper, bold symbol stand for vectors. Given $\mathbf{a}\in\mathbb{R}^3$ we denote by $\mathbf{a}\otimes\mathbf{a}$ the symmetric matrix with entries a_ia_j. Coordinates $\mathbf{x}=(x_1,x_2,x_3)$ are centered on $\mathcal{P}$. For example, we have $\mathcal{P} := \{(x_1,x_2,0),\ (x_1,x_2)\in\mathbb{R}^2\}$. The half-space above $\mathcal{P}$ is $\mathbb{R}^3_+$ and $\mathbb{R}^3_{++}$

stands for $\{(x_1,x_2,x_3)$ with $x_3>1\}$. This is the domain where the center of mass $\mathbf{G}$ can evolve as long as no collision between $\mathcal{B}$ and $\mathcal{P}$ occurs.

Given $\mathbf{G}\in\mathbb{R}^3$, and $\delta>0$, we denote by $\mathcal{B}(\mathbf{G},\delta)$ the sphere with center $\mathbf{G}$ and radius δ. For short, we also set $\mathcal{B}_{\mathbf{G}}=\mathcal{B}(\mathbf{G},1)$. This is the domain occupied by $\mathcal{B}$ when its center of mass meets $\mathbf{G}$. In this case, the fluid domain $\mathcal{F}_{\mathbf{G}}$ is the complementary of $\mathcal{B}_{\mathbf{G}}$ in $\mathbb{R}^3_+$. If the orthogonal projection of $\mathbf{G}$ on $\mathcal{P}$ is the center of coordinates, we have $\mathbf{G}=\mathbf{G}_h=(0,0,1+h)$ with $h=\mathrm{dist}(\mathcal{B}_{\mathbf{G}},\mathcal{P})$. In this case the suitable parameter is h and not $\mathbf{G}$. Thus, when using notations with h as subscript instead of $\mathbf{G}$ we implicitly mean that the subscript should be $\mathbf{G}_h$. For example, $\mathcal{B}_h:=\mathcal{B}_{\mathbf{G}_h}$.

We introduce (r,θ,z) the cylindrical coordinates associated to (x_1,x_2,x_3) :

$$x_1=r\cos(\theta),\quad x_2=r\sin(\theta),\quad x_3=z.$$

Given $h>0$ and $l>0$, we denote by $\Omega_{h,l}$ the symmetric domain under $\mathcal{B}_h$ with width $2l$:

$$\Omega_{h,l}:=\{(r,\theta,z)\in\mathcal{F}_h \text{ such that } r\in[0,l),z\in(0,1+h)\}. \tag{1.5}$$

We notice that, whenever $l<1$, the upper boundary of $\Omega_{h,\delta}$ is parametrized by (r,θ) :

$$(r,\theta,z)\in\partial\Omega_{h,l}\cap\partial\mathcal{B}_h\Leftrightarrow\{r\in[0,\delta) \text{ and } z=\delta_h(r)\}\,,$$

where, for arbitrary non-negative h:

$$\delta_h(s):=1+h-\sqrt{1-s^2},\quad \forall\, s\in(-1,1). \tag{1.6}$$

In the whole paper, we denote by $\eta:[0,\infty)\to[0,1]$ a smooth function such that

$$\eta(s)=\begin{cases}1, & \text{if } s<\frac{1}{2},\\ 0, & \text{if } s>1,\end{cases}$$

and, we set $\eta_\alpha=\eta(\cdot/\alpha)$ for all parameter $\alpha>0$.

We use the classical Lebesgue and Sobolev spaces $L^\alpha(A)$, $W^{\beta,\alpha}(A)$, $H^\beta(A)$ with A an open set, $\alpha\geqslant 1$ and $\beta\geqslant 0$. We define

$$\mathcal{H}=\{\boldsymbol{\phi}\in L^2(\mathbb{R}^3_+)\ ;\ \nabla\cdot\boldsymbol{\phi}=0,\ \boldsymbol{\phi}\cdot\mathbf{n}=0 \text{ on } \mathcal{P}\},$$

$$\mathcal{V}=\{\boldsymbol{\phi}\in H^1(\mathbb{R}^3_+)\ ;\ \nabla\cdot\boldsymbol{\phi}=0,\ \boldsymbol{\phi}=0 \text{ on } \mathcal{P}\},$$

and, for an open subset $A\subset\mathbb{R}^3_+$

$$\mathbb{H}(A)=\{\boldsymbol{\phi}\in\mathcal{H}\ ;\ D(\boldsymbol{\phi})=0 \text{ in } A\},$$

$$\mathbb{V}(A)=\{\boldsymbol{\phi}\in\mathcal{V}\ ;\ D(\boldsymbol{\phi})=0 \text{ in } A\}.$$

To simplify, if $\mathbf{G}\in\mathbb{R}^3_{++}$, we set

$$\mathbb{H}(\mathbf{G})=\mathbb{H}(\mathcal{B}_{\mathbf{G}}),\quad \mathbb{V}(\mathbf{G})=\mathbb{V}(\mathcal{B}_{\mathbf{G}}).$$

For all $\mathbf{G}\in\mathbb{R}^3_{++}$, we also denote by $\rho_{\mathbf{G}}$ the function

$$\rho_{\mathbf{G}}(\mathbf{x})=\begin{cases}1 & \text{if } \mathbf{x}\in\mathcal{B}_{\mathbf{G}},\\ \rho_{\mathcal{B}} & \text{if } \mathbf{x}\in\mathcal{F}_{\mathbf{G}}.\end{cases}$$

If $\mathbf{v}\in\mathbb{H}(\mathbf{G})$, from [13, p.18], there exist $\mathbf{V}_{\mathbf{v}},\boldsymbol{\omega}_{\mathbf{v}}\in\mathbb{R}^3$ such that

$$\mathbf{v}|_{\mathcal{B}_{\mathbf{G}}}=\mathbf{V}_{\mathbf{v}}+\boldsymbol{\omega}_{\mathbf{v}}\times(\mathbf{x}-\mathbf{G}).$$

In particular, if $\mathbf{u},\ \mathbf{v}\in\mathbb{H}(\mathbf{G})$,

$$\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{v}\ d\mathbf{x}=\int_{\mathbb{R}^3_+\setminus\mathcal{B}_{\mathbf{G}}}\mathbf{u}\cdot\mathbf{v}\ d\mathbf{x}+m\mathbf{V}_{\mathbf{u}}\cdot\mathbf{V}_{\mathbf{v}}+J\boldsymbol{\omega}_{\mathbf{u}}\cdot\boldsymbol{\omega}_{\mathbf{v}}.$$

2. Cauchy theory. The aim of this section is to state and prove the existence of weak solutions of (FSIS). We first give the definition of weak solution of (FSIS):

DEFINITION 2.1. *Given $\mathbf{G}^0 \in \mathbb{R}^3_{++}$ and $\mathbf{u}^0 \in \mathbb{H}(\mathbf{G}^0)$, a pair $(\mathbf{u}, \mathbf{G})$ is called weak solution to (FSIS) on $(0,T)$ with initial data $(\mathbf{u}^0, \mathbf{G}^0)$ if*

$$\mathbf{G} \in W^{1,\infty}([0,T]; \mathbb{R}^3_{++}), \quad \textit{with} \quad \mathbf{G}(0) = \mathbf{G}^0, \tag{2.1}$$

$$\mathbf{u} \in L^\infty(0,T; L^2(\mathbb{R}^3_+)) \cap L^2(0,T; H^1_0(\mathbb{R}^3_+)), \tag{2.2}$$

with

$$\mathbf{u} = \mathbf{V} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}) \quad \textit{in } \mathcal{B}_{\mathbf{G}}, \quad \mathbf{V} = \dot{\mathbf{G}},$$

and if

$$-\int_0^T \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \mathbf{u} \cdot \partial_t \mathbf{v} \; d\mathbf{y} \; dt + 2\mu \int_0^T \int_{\mathbb{R}^3_+} D(\mathbf{u}) : D(\mathbf{v}) \; d\mathbf{y} \; dt$$
$$+ \int_0^T \int_{\mathbb{R}^3_+} \mathbf{u} \otimes \mathbf{u} : D(\mathbf{v}) \; d\mathbf{y} \; dt = \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} \left[\mathbf{u}^0 \cdot \mathbf{v}(0) \right] \; d\mathbf{y} \tag{2.3}$$

for all $\mathbf{v} \in \mathcal{C}(0,T; H^1_0(\mathbb{R}^3_+)) \cap H^1(0,T; L^2(\mathbb{R}^3_+))$ with compact support in $(0,T) \times \mathbb{R}^3_+$ and such that $\mathbf{v} \in \mathbb{V}(\mathbf{G}(t))$ for all $t \in [0,T]$,
and if

$$\frac{1}{2} \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} |\mathbf{u}|^2 \; d\mathbf{x} + 2\mu \int_0^t \int_{\mathbb{R}^3_+} |D(\mathbf{u})|^2 \; d\mathbf{x} \leqslant \frac{1}{2} \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} |\mathbf{u}^0|^2 \; d\mathbf{x} \quad \textit{for a.a. } t \in (0,T).$$

The result of well-posedness we obtain for (FSIS) can be stated as follows:

THEOREM 2.2. *Assume $\mathbf{G}^0 \in \mathbb{R}^3_{++}$ and $\mathbf{u}^0 \in \mathbb{H}(\mathbf{G}^0)$, there exists at least one maximal weak solution $(T_0, (\mathbf{U}, \mathbf{G}))$ to (FSIS) with initial data $(\mathbf{U}^0, \mathbf{G}^0)$. Moreover, we have the alternative:*

- $T_0 = \infty$,
- $T_0 < \infty$ *and* $G_3(t) \to 1$ *as* $t \to T_0$.

The proof of Theorem 2.2 we give in the sequel is inspired by methods developed in other papers and since we use many similar arguments, we choose to present here only the main ideas and refer to the appropriate references to avoid repeating technical calculations which are not the main interest of this paper.

2.1. Strong solutions for an approximate system. As in [6], we prove the existence of weak solutions by first obtaining strong solutions for an approximate problem of (FSIS). More precisely, we consider an even non-negative function $\kappa \in C^\infty_0(\mathbb{R})$ such that $\kappa(s) = 0$ if $|s| \geqslant 1$. We define for all $\varepsilon > 0$,

$$K_{1\varepsilon}(t) = c_1 \frac{1}{\varepsilon} \kappa\left(\left| \frac{t}{\varepsilon} \right|^2 \right) \quad (t \in \mathbb{R}), \quad K_{2\varepsilon}(\mathbf{x}) = c_2 \frac{1}{\varepsilon^3} \kappa\left(\left| \frac{\mathbf{x}}{\varepsilon} \right|^2 \right) \quad (\mathbf{x} \in \mathbb{R}^3),$$

and

$$K_\varepsilon(t, \mathbf{x}) = K_{1\varepsilon}(t) K_{2\varepsilon}(\mathbf{x}) \quad (t \in \mathbb{R}, \; \mathbf{x} \in \mathbb{R}^3).$$

The constants c_1 and c_2 are taken in order to satisfy the following relations:

$$\int_{-\infty}^{\infty} K_{1\varepsilon}(t) \; dt = 1 \quad \text{and} \quad \int_{\mathbb{R}^3} K_{2\varepsilon}(\mathbf{x}) \; d\mathbf{x} = 1.$$

Then, for all $\mathbf{u} \in L^2((0,T)\times\mathbb{R}^3_+)$ and for all $\varepsilon > 0$, we set

$$\mathbf{u}_\varepsilon(t,\mathbf{x}) = \int_{(0,T)\times\mathbb{R}^3_+} K_\varepsilon(t-s,\mathbf{x}-\mathbf{y})\mathbf{u}(s,\mathbf{y})\ ds\ d\mathbf{y}.$$

Let us consider the following problem, which approximates (FSIS).

$$\partial_t \mathbf{u} - \mu\Delta\mathbf{u} + (\mathbf{u}_\varepsilon\cdot\nabla)\mathbf{u} + \nabla p = 0 \text{in } \mathcal{F}_{\mathbf{G}(t)},\ t\in(0,T), \tag{2.4}$$

$$\operatorname{div}\mathbf{u} = 0 \text{in } \mathcal{F}_{\mathbf{G}(t)},\ t\in(0,T), \tag{2.5}$$

$$\mathbf{u} = 0 \text{on } \mathcal{P},\ t\in(0,T), \tag{2.6}$$

$$\mathbf{u} = \dot{\mathbf{G}} + \boldsymbol{\omega}\times(\mathbf{x}-\mathbf{G}) \text{on } \partial\mathcal{B}_{\mathbf{G}},\ t\in(0,T), \tag{2.7}$$

$$m\ddot{\mathbf{G}} = -\int_{\partial\mathcal{B}_{\mathbf{G}}} \mathbb{T}(\mathbf{u},p)\mathbf{n}\ d\sigma + \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}}} ((\mathbf{u}_\varepsilon-\mathbf{u})\cdot\mathbf{n})\mathbf{u}\ d\sigma \quad \text{in } (0,T), \tag{2.8}$$

$$J\dot{\boldsymbol{\omega}} = -\int_{\partial\mathcal{B}_{\mathbf{G}}} (\mathbf{x}-\mathbf{G})\times\mathbb{T}(\mathbf{u},p)\mathbf{n}\ d\sigma + \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}}} ((\mathbf{u}_\varepsilon-\mathbf{u})\cdot\mathbf{n})(\mathbf{x}-\mathbf{G})\times\mathbf{u}\ d\sigma \quad \text{in } (0,T). \tag{2.9}$$

We complete the system with the initial conditions

$$\mathbf{u}(0,\cdot) = \mathbf{u}^0,\quad \mathbf{V}(0) = \mathbf{V}^0,\quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}^0,\quad \mathbf{G}(0) = \mathbf{G}^0. \tag{2.10}$$

We define the space $\widehat{H^1}(A)$ by

$$\widehat{H^1}(A) = \{q\in L^2_{loc}(\overline{A})\ ;\ \nabla q\in L^2(A)\},$$

We denote

$$\mathcal{F}_T = \{(t,\mathbf{x})\in[0,T]\times\mathbb{R}^3\ ;\ \mathbf{x}\in\mathcal{F}_{\mathbf{G}(t)}\}.$$

Consider a smooth mapping $\mathbf{X} : \mathbb{R}^3_{++}\times\mathcal{F}_{\mathbf{G}^0}\to\mathbb{R}^3$ such that for all $\mathbf{G}\in\mathbb{R}^3_{++}$, $\mathbf{X}(\mathbf{G},\cdot)$ is a C^∞-diffeomorphism from $\mathcal{F}_{\mathbf{G}^0}$ onto $\mathcal{F}_{\mathbf{G}}$. Moreover, suppose that the mappings

$$(\mathbf{G},\mathbf{y})\mapsto D_{\mathbf{G}}D^\alpha_y\mathbf{X}(\mathbf{G},\mathbf{y}),\quad \alpha\in\mathbb{N}^3,$$

exist, are continuous and compactly supported in $\mathcal{F}_{\mathbf{G}^0}$. For any $\mathbf{g} : \mathcal{F}_T\to\mathbb{R}^3$, we denote by $\mathbf{g}_{\mathbf{X}} : [0,T]\times\mathcal{F}_{\mathbf{G}^0}\to\mathbb{R}^3$ the mapping $\mathbf{g}_{\mathbf{X}}(t,\mathbf{y}) = \mathbf{g}(t,\mathbf{X}(\mathbf{G}(t),\mathbf{y}))$, for all $t\geqslant 0$ and for all $\mathbf{y}\in\mathcal{F}_{\mathbf{G}^0}$. We use a similar notation for $g : \mathcal{F}_T\to\mathbb{R}$.

We introduce the following functions spaces in variable domain:

$$\begin{aligned} L^2(0,T;H^2(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}}\in L^2(0,T;H^2(\mathcal{F}_{\mathbf{G}^0}))\},\\ H^1(0,T;L^2(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}}\in H^1(0,T;L^2(\mathcal{F}_{\mathbf{G}^0}))\},\\ C([0,T],H^1(\mathcal{F}(t))) &= \{\mathbf{u}\ ;\ \mathbf{u}_{\mathbf{X}}\in C([0,T],H^1(\mathcal{F}_{\mathbf{G}^0}))\},\\ L^2(0,T;\widehat{H}^1(\mathcal{F}(t))) &= \{p\ ;\ p_{\mathbf{X}}\in L^2(0,T;\widehat{H}^1(\mathcal{F}_{\mathbf{G}^0}))\}. \end{aligned}$$

Theorem 2.3. *Assume that*

$$\operatorname{dist}(\mathbf{G}^0,\mathcal{P}) = d^0 > 0$$

Assume moreover that $\mathbf{u}^0\in H^1(\mathcal{F})$ *and satisfies*

$$\nabla\cdot\mathbf{u}^0 = 0 \text{ in } \mathcal{F},\quad \mathbf{u}^0 = 0 \text{ on } \mathcal{P},\quad \mathbf{u}^0(\mathbf{y}) = \mathbf{V}^0 + \boldsymbol{\omega}^0\times(\mathbf{y}-\boldsymbol{G}^0) \text{ on } \partial\mathcal{B}_{\mathbf{G}^0}.$$

Then there exists a time $T > 0$ and a strong solution of (2.4)–(2.9) *on* $[0, T]$ *such that*

$$u \in L^2(0,T;H^2(\mathcal{F}(t))) \cap C([0,T];H^1(\mathcal{F}(t))) \cap H^1(0,T;L^2(\mathcal{F}(t))), \quad p \in L^2(0,T;\widehat{H^1}(\mathcal{F}(t))),$$

$$\mathbf{G} \in H^2(0,T), \quad \boldsymbol{\omega} \in H^1(0,T).$$

The time T can be chosen such that

$$\operatorname{dist}(\mathbf{G}(t), \mathcal{P}) = \frac{d^0}{2} > 0 \quad \forall t \in [0,T] \tag{2.11}$$

and $T = T(\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)})$.

Proof. To prove the local in time existence of solution of system (2.4)–(2.9), we can consider the method used in [2, 3, 12]. The idea is to consider a change of variables which is diffeomorphism of $\mathbb{R}^3_+$ which transforms $\mathcal{F}$ onto $\mathcal{F}(t)$ (see also [9]) and to study the system obtained after the change of variables.

To obtain the global in time existence (until a possible contact), we derive a priori estimates: first, we multiply (2.4) by $\mathbf{u}$, (2.8) by $\dot{\mathbf{G}}$ and (2.9) by $\boldsymbol{\omega}$. We deduce the energy estimate

$$\frac{1}{2}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}(t)} |\mathbf{u}|^2(t)\ d\mathbf{x} + 2\mu \int_0^t \int_{\mathbb{R}^3_+} |D(\mathbf{u})|^2\ d\mathbf{x} = \frac{1}{2}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} |\mathbf{u}^0|^2\ d\mathbf{x}.$$

From the above estimate we can obtain a time $T > 0$ depending on $\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^0} |\mathbf{u}^0|^2\ d\mathbf{x}$ such that (2.11) holds.

Then we consider a smooth function Υ with compact support in $\mathcal{B}(\mathbf{G}^0, 1+\delta/4)$ and such that $\Upsilon = 1$ on $\mathcal{B}_{\mathbf{G}^0}$. We also set

$$\mathbf{w}_R = \frac{1}{2}\left(\dot{\mathbf{G}} \times (\mathbf{x} - \mathbf{G}) + |\mathbf{x} - \mathbf{G}|^2 \boldsymbol{\omega}\right)$$

We then define

$$\mathbf{u}_R(t,\mathbf{x}) = \operatorname{curl}\left(\Upsilon\left(\mathbf{x} - \mathbf{G}(t) + \mathbf{G}^0\right)\mathbf{w}(t,\mathbf{x})\right).$$

This function satisfies

$$\nabla \cdot \mathbf{u}_R = 0, \quad \mathbf{u}_R = \dot{\mathbf{G}} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{G}) \quad \text{on } \mathcal{B}_{\mathbf{G}(t)}.$$

Then we multiply (2.4) by

$$\boldsymbol{\varphi} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}_R \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u}_R,$$

and by using the regularization of the nonlinear term in (2.4), we obtain that the mapping

$$t \mapsto \|\mathbf{u}\|_{H^1(\mathcal{F}(t))}$$

is bounded on $[0, T]$. We refer to [2, 3] for details of the previous derivations. □

2.2. Convergences. Let $\mathbf{u}^0$ be as in Theorem 2.2. There exists a sequence $\mathbf{u}^{0k} \in H^1(\mathcal{F})$ satisfying

$$\operatorname{div} \mathbf{u}^{0k} = 0 \quad \text{in } \mathbb{R}^3_+, \quad D(\mathbf{u}^{0k}) = 0 \quad \text{in } \mathcal{B}_{\mathbf{G}^0}, \quad \mathbf{u}^{0k} = 0 \quad \text{on } \mathcal{P}$$

and such that

$$\mathbf{u}^{0k} \to \mathbf{u}^0 \quad \text{in } L^2.$$

We also take a sequence $\varepsilon^k \to 0$. There exists a time T such that for all k, the corresponding solutions $\mathbf{u}^k$ exist on $[0,T]$ and

$$\operatorname{dist}(\mathbf{G}^k(t),\mathcal{P}) > \frac{d^0}{2} \quad \forall\, t \in [0,T],\ \forall\, k.$$

$$\mathbf{u}^k \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T;H^1(\mathbb{R}^3_+))\text{-weak and } L^\infty(0,T;L^2(\mathbb{R}^3_+))\text{-weak}^*, \tag{2.12}$$

$$\mathbf{G}^k \to \mathbf{G} \quad \text{in } C([0,T];\mathbb{R}^3_{++}). \tag{2.13}$$

Taking any $\mathbf{U} \in \mathcal{D}((0,T)\times\mathbb{R}^3_+)$ such that $D(\mathbf{U}) = 0$ in a neighborhood of $\mathcal{B}_{\mathbf{G}(t)}$ for all $t \in (0,T)$, we can multiply (2.4) by $\mathbf{U}$ for k sufficiently large. After integration by parts, it yields:

$$\begin{aligned}
-\int_0^T\int_{\mathbb{R}^2_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\partial_t\mathbf{U} + \int_0^T\int_{\mathbb{R}^3_+}(\mathbf{u}^k\otimes\mathbf{u}^k):D(\mathbf{U}) + 2\mu\int_0^T\int_{\mathbb{R}^3_+} D(\mathbf{u}^k):D(\mathbf{U})\,d\mathbf{x}\\
= \int_0^T\int_{\mathcal{F}^k}\left[((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\nabla)\mathbf{U}\right]\cdot\mathbf{u}^k - \frac{1}{2}\int_0^T\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}(\mathbf{U}\cdot\mathbf{u}^k)((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\mathbf{n})\,d\sigma
\end{aligned} \tag{2.14}$$

As classical, the main difficulty in order to pass to the limit here is to prove L^2-compactness of the $\mathbf{u}^k$. The procedure to prove this compactness property follows closely the method developed in [6].

As a first step, we divide the segment $[0,T]$ into N segments $[t_{i-1},t_i]$, with $\Delta t = t_i - t_{i-1} = T/N$, $i = 1,\dots,N$. For all i and for $\delta < \frac{d^0}{2}$, we consider an orthonormal basis $(\mathbf{e}^{i,\delta}_j)$ of $\mathbb{H}(\mathcal{B}(\mathbf{G}(t_i),1+\delta))$. Without further restriction, we assume all the $\mathbf{e}^{i,\delta}_j$ with compact support. We also consider the set of piecewise linear functions in t

$$\mathbf{U}^\delta(t,\mathbf{x}) = \mathbf{e}^{i-1,\delta}_j(\mathbf{x}) + \frac{t-t_i}{\Delta t}\left(\mathbf{e}^{i,\delta}_l(\mathbf{x}) - \mathbf{e}^{i-1,\delta}_j(\mathbf{x})\right), \tag{2.15}$$

for $t\in[t_{i-1},t_i]$, $j,l\in\mathbb{N}$, $i\in\{1,\dots,N\}$. There exists a countable set of functions satisfying (2.15).

From (2.13), there exists $k_0 = k_0(\delta)$ such that for $k \geqslant k_0$,

$$D(\mathbf{U}^\delta) = 0 \quad \text{in } \mathcal{B}_{\mathbf{G}^k(t)}, \quad \forall t\in[0,T]$$

for all functions satisfying (2.15).

We multiply (2.4) by $\mathbf{U}^\delta$ satisfying (2.15):

$$\begin{aligned}
\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{u}^k}{\partial t} + (\mathbf{u}^k_\varepsilon\cdot\nabla)\mathbf{u}^k\right)\cdot\mathbf{U}^\delta\,d\mathbf{x} + 2\mu\int_{\mathcal{F}^k(t)} D(\mathbf{u}^k):D(\mathbf{U}^\delta)\,d\mathbf{x}\\
-\mathbf{V}_{\mathbf{U}^\delta}\cdot\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}\mathbb{T}(\mathbf{u}^k,p^k)\mathbf{n}\,d\sigma - \boldsymbol{\omega}_{\mathbf{U}^\delta}\cdot\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}(\mathbf{x}-\mathbf{G}^k)\times\mathbb{T}(\mathbf{u}^k,p^k)\mathbf{n}\,d\sigma.
\end{aligned} \tag{2.16}$$

We notice that

$$\begin{aligned}
\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{u}^k}{\partial t} + (\mathbf{u}^k_\varepsilon\cdot\nabla)\mathbf{u}^k\right)\cdot\mathbf{U}^\delta\,d\mathbf{x} = \frac{d}{dt}\int_{\mathcal{F}^k(t)}\mathbf{u}^k\cdot\mathbf{U}^\delta\,d\mathbf{x}\\
-\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{U}^\delta}{\partial t} + (\mathbf{u}^k\cdot\nabla)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\,d\mathbf{x} + \int_{\mathcal{F}^k(t)}\left(\left((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\nabla\right)\mathbf{u}^k\right)\cdot\mathbf{U}^\delta\,d\mathbf{x}.
\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) and using (2.8) and (2.9) we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta\ d\mathbf{x} = -2\mu\int_{\mathcal{F}^k(t)} D(\mathbf{u}^k):D(\mathbf{U}^\delta)\ d\mathbf{x}$$
$$+\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{U}^\delta}{\partial t}+(\mathbf{u}^k\cdot\nabla)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\ d\mathbf{x} - \int_{\mathcal{F}^k(t)}\left(\left((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\nabla\right)\mathbf{u}^k\right)\cdot\mathbf{U}^\delta\ d\mathbf{x}$$
$$+\frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}(\mathbf{U}^\delta\cdot\mathbf{u}^k)((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\mathbf{n})\ d\sigma. \tag{2.18}$$

Integrating by parts in the above relation yields

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta\ d\mathbf{x} = -2\mu\int_{\mathcal{F}^k(t)} D(\mathbf{u}^k):D(\mathbf{U}^\delta)\ d\mathbf{x}$$
$$+\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{U}^\delta}{\partial t}+(\mathbf{u}^k\cdot\nabla)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\ d\mathbf{x} + \int_{\mathcal{F}^k(t)}\left(\left((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\nabla\right)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\ d\mathbf{x}$$
$$-\frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}(\mathbf{U}^\delta\cdot\mathbf{u}^k)((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\mathbf{n})\ d\sigma \tag{2.19}$$

and thus

$$\frac{d}{dt}\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta\ d\mathbf{x} = -2\mu\int_{\mathcal{F}^k(t)} D(\mathbf{u}^k):D(\mathbf{U}^\delta)\ d\mathbf{x}$$
$$+\int_{\mathcal{F}^k(t)}\left(\frac{\partial\mathbf{U}^\delta}{\partial t}+(\mathbf{u}^k_\varepsilon\cdot\nabla)\mathbf{U}^\delta\right)\cdot\mathbf{u}^k\ d\mathbf{x} - \frac{1}{2}\int_{\partial\mathcal{B}_{\mathbf{G}^k(t)}}(\mathbf{U}^\delta\cdot\mathbf{u}^k)((\mathbf{u}^k_\varepsilon-\mathbf{u}^k)\cdot\mathbf{n})\ d\sigma \tag{2.20}$$

Following the estimates of [6], using Arzela and Ascoli and the diagonal Cantor procedure, we obtain that for all $\mathbf{U}^\delta$ satisfying (2.15), we have

$$\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}^\delta\ d\mathbf{x}\to\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}^\delta\ d\mathbf{x}\quad\text{in } C([0,T]). \tag{2.21}$$

Then, we can fix the position of $\mathbf{G}$ using the same change of variable as in the proof of Theorem 2.3 and apply similar arguments to the proof of Lemmas 4.1 and 4.2 of [6] to deduce that for all $\mathbf{U}\in L^2(0,T;L^2(\mathbb{R}^3_+))$, $\nabla\cdot\mathbf{U}=0$, $D(\mathbf{U})=0$ in $\mathcal{B}(\mathbf{G}(t),1)$,

$$\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}^k}\mathbf{u}^k\cdot\mathbf{U}\ d\mathbf{x}\to\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}\ d\mathbf{x}\quad\text{in } L^2(0,T). \tag{2.22}$$

However, as $\mathbf{G}^k\to\mathbf{G}$ in $\mathcal{C}([0,T])$, and $\mathbf{u}^k$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^3_+))$, this leads also to:

$$\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}}\mathbf{u}^k\cdot\mathbf{U}\ d\mathbf{x}\to\int_{\mathbb{R}^3_+}\rho_{\mathbf{G}}\mathbf{u}\cdot\mathbf{U}\ d\mathbf{x}\quad\text{in } L^2([0,T]). \tag{2.23}$$

We extend $\mathbf{u}^k$ by 0 in $\mathbb{R}^3\setminus\mathbb{R}^3_+$, and since $\mathbf{u}^k=0$ on $\mathcal{P}$, we have $\mathbf{u}^k\in L^2(0,T;H^1(\mathbb{R}^3))$. We also set

$$\widehat{\mathbf{u}}^k(t,\mathbf{y})=\mathbf{u}^k(t,\mathbf{y}+\mathbf{G}^k(t)-\mathbf{G}(t)).$$

We have

$$\nabla\cdot\widehat{\mathbf{u}}^k=0,\quad \widehat{\mathbf{u}}^k=0\quad\text{on } \{\mathbf{y}\in\mathbb{R}^3\ ;\ y_3=h-h^k\},\quad D(\mathbf{u}^k)=0\quad\text{in } \mathcal{B}_{\mathbf{G}}$$

and

$$\|\widehat{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \leqslant \|\mathbf{G}^k - \mathbf{G}\|_{L^\infty(0,T)} \|\nabla \mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \to 0. \tag{2.24}$$

We define

$$\mathcal{L}^2(\mathbb{R}^3) = \left\{\mathbf{v} \in L^2(\mathbb{R}^3)\ ;\ \operatorname{div}\mathbf{v} = 0 \quad \text{in } \mathbb{R}^3\right\}.$$

Following [6], there exists a function $\Lambda : \mathcal{L}^2(\mathbb{R}^3) \to \mathcal{L}^2(\mathbb{R}^3)$ such that for all $\mathbf{v} \in \mathcal{L}^2(\mathbb{R}^3)$, $\mathbf{u} = \Lambda(\mathbf{v})$ satisfies

$$\mathbf{u} = \mathbf{v} \quad \text{in } \mathcal{B}_{\mathbf{G}^0}, \quad \mathbf{u} = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\mathcal{B}(\mathbf{G}^0, 1 + \frac{d^0}{4})},$$

$$\|\mathbf{u}\|_{L^2(\mathbb{R}^3)} \leqslant c\|\mathbf{v}\|_{L^2(\mathbb{R}^3)}.$$

Moreover,

$$\text{if} \quad \mathbf{v} \in C([0,T]; L^2(\mathbb{R}^3)), \quad \text{then} \quad \Lambda\mathbf{v} \in C([0,T]; L^2(\mathbb{R}^3))$$

and

$$\text{if} \quad \mathbf{v} \in L^2(0,T; L^2(\mathbb{R}^3)), \quad \text{then} \quad \Lambda\mathbf{v} \in L^2(0,T; L^2(\mathbb{R}^3)).$$

We also define $\check{\Lambda}^t : \mathcal{L}^2(\mathbb{R}^3) \to \mathcal{L}^2(\mathbb{R}^3)$ as follows:

$$\check{\Lambda}^t \check{\mathbf{v}}(\mathbf{y}) = [\Lambda\check{\mathbf{v}}]\,(\mathbf{y} + \mathbf{G}^0 - \mathbf{G}(t)),$$

where

$$\check{\mathbf{v}}(\mathbf{x}) = \mathbf{v}(\mathbf{x} + \mathbf{G}(t) - \mathbf{G}^0).$$

Then we consider as in [6]:

$$\overline{\mathbf{u}}^k(\mathbf{x},t) = \mathbf{u}^k + \check{\Lambda}^t(\widehat{\mathbf{u}}^k - \mathbf{u}^k).$$

This function is rigid in $\mathcal{B}_{\mathbf{G}(t)}$ and

$$\|\overline{\mathbf{u}}^k - \mathbf{u}^k\|_{L^2((0,T)\times\mathbb{R}^3)} \to 0. \tag{2.25}$$

Combining (2.23), (2.25) and (2.13) we obtain for all $\mathbf{U} \in L^2(0,T; L^2(\mathbb{R}^3_+))$, $\nabla \cdot \mathbf{U} = 0$, $D(\mathbf{U}) = 0$ in $\mathcal{B}(\mathbf{G}(t), 1)$,

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}}^k \cdot \mathbf{U}\ d\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \mathbf{u} \cdot \mathbf{U}\ d\mathbf{x} \quad \text{in } L^2(0,T). \tag{2.26}$$

We are now in position to apply Friedrichs Lemma (see [5], Lemma II.4.2): for all $\mathcal{O} \subset \mathbb{R}^3_+$, for any $\gamma > 0$ there exist $I = I(\gamma, \mathcal{O}) \in \mathbb{N}$ and functions $\boldsymbol{\psi}_j \in L^\infty(\mathcal{O})$, $j = 1, \dots, I$ such that

$$\begin{aligned} &\left\|\mathbf{u}^k - \mathbf{u}\right\|^2_{L^2((0,T);L^2(\mathcal{O}))} \\ &\qquad \leqslant \sum_{j=1}^I \int_0^T \left(\int_{\mathcal{O}} \left(\mathbf{u}^k(t) - \mathbf{u}(t)\right) \cdot \boldsymbol{\psi}_j\ \mathrm{d}\mathbf{y}\right)^2\ \mathrm{d}t + \gamma \left\|\nabla\mathbf{u}^k - \nabla\mathbf{u}\right\|^2_{L^2(0,T;L^2(\mathcal{O}))}. \end{aligned} \tag{2.27}$$

Assume that $\boldsymbol{\psi} \in L^\infty(\mathcal{O})$ with

$$\mathcal{B}_{\mathbf{G}^k(t)} \subset \mathcal{O} \quad \forall t \in [0,T], \forall k > k_0$$

(this is possible by using (2.13)).

We consider an orthonormal basis $(\boldsymbol{\phi}_i)_{i\in\mathbb{N}}$ of $\mathcal{H}$. We have the following extension:

$$\boldsymbol{\psi} = \sum_{i=1}^{\infty} \alpha_i \boldsymbol{\phi}_i + \boldsymbol{\xi},$$

with $\boldsymbol{\xi} \in \mathcal{H}^{\perp}$.

We denote by $P(\mathbf{G})$ the orthogonal projection from $\mathcal{H}$ onto $\mathbb{H}(\mathbf{G})$. Let us also set

$$\widetilde{\boldsymbol{\phi}}_i = P(\mathbf{G})\boldsymbol{\phi}_i.$$

We notice that $\widetilde{\boldsymbol{\phi}}_i \in L^2(0,T;L^2(\mathbb{R}^3_+))$, $\nabla \cdot \widetilde{\boldsymbol{\phi}}_i = 0$, $D(\widetilde{\boldsymbol{\phi}}_i) = 0$ in $\mathcal{B}(\mathbf{G}(t),1)$ and that

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}}^k \cdot \widetilde{\boldsymbol{\phi}}_i = \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}}^k \cdot \boldsymbol{\phi}_i \quad \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}} \cdot \widetilde{\boldsymbol{\phi}}_i = \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}} \cdot \boldsymbol{\phi}_i.$$

We take $\mathbf{U} = \widetilde{\boldsymbol{\phi}}_i$ in (2.26) and by using the energy estimate and the above relations, we obtain (2.26) for $\mathbf{U} = \sum_{i=1}^{\infty} \alpha_i \boldsymbol{\phi}_i$. For $\mathbf{U} = \boldsymbol{\xi}$, both side of (2.26) is equal to 0. As a consequence,

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \overline{\mathbf{u}}^k \cdot \boldsymbol{\psi} \, d\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \mathbf{u} \cdot \boldsymbol{\psi} \, d\mathbf{x} \quad \text{in } C([0,T]). \tag{2.28}$$

From (2.25) and the above equation we obtain

$$\int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \mathbf{u}^k \cdot \boldsymbol{\psi} \, d\mathbf{x} \to \int_{\mathbb{R}^3_+} \rho_{\mathbf{G}} \mathbf{u} \cdot \boldsymbol{\psi} \, d\mathbf{x} \quad \text{in } C([0,T]). \tag{2.29}$$

The above relation and (2.27) yield

$$\mathbf{u}^k \to \mathbf{u} \quad \text{in } L^2(\mathcal{O}).$$

Using the above relation, we can pass to the limit in (2.14) and we obtain the weak formulation (2.3) for smooth test-functions. We can pass from smooth test-functions to the required regularity for $\mathbf{v}$ applying the same approximation technique as when we obtained (2.23).

3. Constructing test-functions. As in [8], we estimate the distance between $\mathcal{B}$ and $\mathcal{P}$ from below with a suitable choice of test-function in the weak formulation. To this end, we introduce an approximation of the Stokes solution for a given position of $\mathcal{B}$ in $\mathbb{R}^3_+$ (namely $\mathcal{B}_h$). We call these approximations "static functions" and denote them by $(\mathbf{w}_h)_{h>0}$. Given a weak solution $(\mathbf{u},\mathbf{G})$ to (FSIS) in $(0,T)$, we construct admissible test-functions setting:

$$\begin{array}{rrcl} \widetilde{\mathbf{w}}: & (0,T)\times\mathbb{R}^3_+ & \longrightarrow & \mathbb{R}^3, \\ & (t,\mathbf{x}) & \longmapsto & \zeta(t)\mathbf{w}_{h(t)}(x_1-G_1(t),x_2-G_2(t),x_3), \end{array} \tag{3.1}$$

for arbitrary $\zeta\in\mathcal{D}(0,T)$. In this definition $h(t)$ stands for the distance between the sphere and the ramp at time t.

Applying the weak formulation, we obtain:

$$\int_0^T\int_{\mathbb{R}^3_+}\left[\rho_{\mathbf{G}}\mathbf{u}\cdot\widetilde{\mathbf{w}}_t+(\mathbf{u}\otimes\mathbf{u}-2\mu\mathrm{D}(\mathbf{u})):\mathrm{D}(\widetilde{\mathbf{w}})\right]\,d\mathbf{x}\,dt=0$$

In this equation, the key ingredient is:

$$\int_{\mathbb{R}^3_+}\mathrm{D}(\mathbf{u}):\mathrm{D}(\mathbf{w}_h).$$

It shall behave like $\dot{h}/h^{\alpha}$ with an exponent α to be made precise. The other terms appear as remainders. We shall bound them by an integrable (in time) function. This relies on:

LEMMA 3.1. *Given $h>0$, $r_0>0$ and $(\mathbf{u},\mathbf{w})\in H^1_0(\mathbb{R}^3_+)\times(\mathbb{H}(\mathbf{G}_h)\cap\mathcal{C}^\infty(\mathcal{F}_h))$, we assume $\mathbf{w}$ is with compact support. Then, there exists C depending only on the size of the support of $\mathbf{w}$ such that:*

$$\left|\int_{\mathbb{R}^3_+}\mathbf{u}\cdot\mathbf{w}\,\,d\mathbf{x}\right|\leqslant C\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^3_+)}\left[\|\mathbf{w}\|_{2,2}+\|\mathbf{w}\|_{L^2\left(\mathbb{R}^3_+\setminus\Omega_{h,r_0}\right)}\right], \tag{3.2}$$

where

$$\|\mathbf{w}\|^2_{2,2}=\int_0^{r_0}\left(\delta_h(r)^2\left[\int_0^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}\{|\mathbf{w}(r,\theta,z)|^2)\}dz\right]\right)r\,dr.$$

If moreover $\mathbf{w}\in\mathbb{V}(\mathbf{G}_h)$, we have

$$\left|\int_{\mathbb{R}^3_+}\mathbf{u}\otimes\mathbf{u}:D(\mathbf{w})\,\,d\mathbf{x}\right|\leqslant C\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3_+)}\left[\|\mathbf{w}\|_{\infty,2}+\|D(\mathbf{w})\|_{L^\infty\left(\mathcal{F}_h\setminus\Omega_{h,r_0}\right)}\right], \tag{3.3}$$

where

$$\|\mathbf{w}\|_{\infty,2}=\sup_{r\in(0,r_0)}\left\{\delta_h(r)^{\frac{3}{2}}\left[\int_0^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}\{|\nabla\mathbf{w}(r,\theta,z)|^2\}dz\right]^{\frac{1}{2}}\right\}.$$

Proof. We denote by I_1 and I_2 the two integrals we want to estimate in (3.3) and (3.2).

We first deal with I_1. As $D(\mathbf{w})=0$ in $\mathcal{B}_h$, we might restrict the integration domain to $\mathcal{F}_h$. We split the integral into an integral in $\mathcal{F}_h\setminus\Omega_{h,r_0}$ and an integral in Ω_{h,r_0}: $I_1=I_1^{in}+I_1^{out}$ with

$$|I_1^{out}|=\left|\int_{\mathcal{F}_h\setminus\Omega_{h,r_0}}\mathbf{u}\otimes\mathbf{u}:D(\mathbf{w})\,\,\mathrm{d}\mathbf{x}\right|\leqslant\|\mathbf{u}\|^2_{L^2(\mathrm{Supp}(\mathbf{w}))}\|D(\mathbf{w})\|_{L^\infty\left(\mathcal{F}_h\setminus\Omega_{h,r_0}\right)}.$$

Because, Supp($\mathbf{w}$) is bounded and $\mathbf{u} \in H^1_0(\mathbb{R}^3_+)$ we can use the Poincaré inequality. Concerning the integral in Ω_{h,r_0}, we have:

$$I_1^{in} = \int_0^{2\pi} \int_0^{r_0} \int_0^{\delta_h(r)} \left[\mathbf{u}(r,\theta,z) \otimes \mathbf{u}(r,\theta,z) : D(\mathbf{w})(r,\theta,z)\right] r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta,$$

Using a Hölder inequality with respect to the z-variable, we deduce

$$|I_1^{in}| \leqslant C \int_0^{2\pi} \int_0^{r_0} \left[\int_0^{\delta_h(r)} |\mathbf{u}(r,\theta,z)|^4 \mathrm{d}z\right]^{\frac12} \left[\int_0^{\delta_h(r)} |D(\mathbf{w})|^2 \mathrm{d}z\right]^{\frac12} r\mathrm{d}r\mathrm{d}\theta.$$

Then, a direct generalization of the Poincaré inequality (see **Lemma 12** in [8]) implies:

$$\left[\int_0^{\delta_h(r)} |\mathbf{u}(r,\theta,z)|^4 \mathrm{d}z\right]^{\frac12} \leqslant C\delta_h(r)^{\frac32} \left[\int_0^{\delta_h(r)} |\nabla \mathbf{u}(r,\theta,z)|^2 \mathrm{d}z\right].$$

Replacing in I_1 and using again a Hölder inequality, we then obtain (3.3).

To estimate I_2, we decompose it in the same manner as I_1, and with same proof, we deduce that there exists $C = C(\mathrm{Supp}(w))$ such that

$$|I_2^{out}| \leqslant C\|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \|\mathbf{w}\|_{L^2\left(\mathbb{R}^3_+ \setminus \Omega_{h,r_0}\right)}.$$

It remains to estimate the integral in Ω_{h,r_0}:

$$I_2^{in} = \int_0^{2\pi} \int_0^{r_0} \int_0^{\delta_h(r)} \left[\mathbf{u}(r,\theta,z) \cdot \mathbf{w}(r,\theta,z)\right] r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

As above, a Hölder inequality in the z-variable associated to the Poincaré inequality implies:

$$|I_2^{in}| \leqslant C \int_0^{2\pi} \int_0^{r_0} \left[\int_0^{\delta_h(r)} |\nabla \mathbf{u}(r,\theta,z)|^2 \mathrm{d}z\right]^{\frac12} \delta_h(r) \left[\int_0^{\delta_h(r)} |\mathbf{w}|^2 \mathrm{d}z\right]^{\frac12} r\mathrm{d}r\mathrm{d}\theta.$$

We conclude by using a Cauchy–Schwarz inequality. □

3.1. Explicit formula. From now on h is a fixed positive parameter. As in [8], we introduce a velocity-field which is a good approximation (in a sense to be made precise) to the Stokes problem:

$$\left\{\begin{array}{rcl} \operatorname{div} \mathbb{T}(\mathbf{w}, q) &=& 0, \\ \operatorname{div} \mathbf{w} &=& 0, \end{array} \quad \text{in } \mathcal{F}_h \right. \\ \begin{array}{l} \mathbf{w}_{|\mathcal{P}} = 0, \\ \mathbf{w}_{|\partial\mathcal{B}_h} = \mathbf{e}_3. \end{array} \tag{3.4}$$

At first, we focus on the divergence-free and boundary conditions. So we introduce a potential vector-field $\mathbf{a}_h$ and set $\mathbf{w}_h = \nabla \times \mathbf{a}_h$. One choice for $\mathbf{a}_h$ could be:

$$\mathbf{a}_h^s(\mathbf{x}) := \frac{\eta_{h_0}(|\mathbf{x} - \mathbf{G}_h| - 1)}{2} (\mathbf{e}_3 \times (\mathbf{x} - \mathbf{G}_h)), \quad \forall\, \mathbf{x} \in \mathcal{F}_h, \quad \text{with } h_0 > 0.$$

The field $\mathbf{w}_h^s := \nabla \times \mathbf{a}_h^s$ satisfies the divergence-free and boundary conditions whatever the value of $h_0 < h$. However, when h goes to 0, this particular velocity-field does not take advantage of the particular shape of the aperture between $\mathcal{B}$ and $\mathcal{P}$. Thus, we need to find another velocity-field, especially in this aperture, in $\Omega_{h,1/2}$.

As we want to obtain an approximation of the Stokes problem, we construct a velocity field in which the fluid escapes from under the sphere with the most efficiency. Consequently, we want the velocity-field to be planar and radial in each plane. Thus, our potential vector field reads, in cylindrical coordinates:

$$\mathbf{a}_h^d(r,\theta,z) = \left(-\phi_h^d(r,z)\sin(\theta), \phi_h^d(r,z)\cos(\theta), 0\right)^\top, \quad \forall\,(r,\theta,z) \in \Omega_{h,1/2},$$

so that, for all $(r,\theta,z) \in \Omega_{h,1/2}$:

$$\mathbf{w}_h^d(r,\theta,z) = \left(-\partial_z\phi_h^d(r,z)\cos(\theta), -\partial_z\phi_h^d(r,z)\sin(\theta), \partial_r\phi_h^d(r,z) + \frac{\phi_h^d(r,z)}{r}\right)^\top.$$

We set, in order to fit boundary conditions (this shall be critical in **Lemma 3.2**):

$$\phi_h^d(r,z) = r\chi_o\left(\frac{z}{\delta_h(r)}\right), \text{ with } \chi_o(s) = \frac{s^2(3-2s)}{2}, \quad \forall\, s \in (0,1).$$

From now on, we set $h_0 = (\sqrt{17/16} - 1)/2$. It remains to interpolate $\mathbf{w}_h^s$ and $\mathbf{w}_h^d$ so that, we obtain:

$$\mathbf{a}_h(\mathbf{x}) = \begin{cases} \eta_{1/2}(r)\mathbf{a}_h^d(r,\theta,z) + (1-\eta_{1/2}(r))\mathbf{a}_h^s(\mathbf{x}), & \text{in } \Omega_{h,1/2}, \\ \mathbf{a}_h^s(\mathbf{x}), & \text{in } \mathbb{R}^3_+ \setminus \Omega_{h,1/2}. \end{cases}$$

and $\mathbf{w}_h = \nabla \times \mathbf{a}_h$. Explicitly, in $\Omega_{h,1/2}$, we have:

$$\mathbf{w}_h(r,\theta,z) = \eta_{1/2}(r)\mathbf{w}_h^d(r,\theta,z) + (1-\eta_{1/2}(r))\mathbf{w}_h^s(\mathbf{x}) + \mathbf{rem}_0(\mathbf{x}), \tag{3.5}$$

where, denoting by $\boldsymbol{n\pi}_3(\mathbf{x}) = (x_1,x_2,0)^\top/\sqrt{x_1^2+x_2^2}$, we have:

$$\mathbf{rem}_0(\mathbf{x}) = \eta'_{1/2}(r)\boldsymbol{n\pi}_3(\mathbf{x}) \times \left(\mathbf{a}_h^d(r,\theta,z) - \mathbf{a}_h^s(\mathbf{x})\right), \quad \text{in } \Omega_{h,1/2}.$$

3.2. From static to moving test-function. The main point in this subsection is to prove that, given a weak solution to (FSIS) $(\mathbf{u},\mathbf{G})$ and $\zeta \in \mathcal{D}(0,T)$, the function $\widetilde{\mathbf{w}}$ constructed in (3.1) is a suitable test-function. To this end, we need to extend $\mathbf{w}_h$ on $\mathbb{R}^3_+$ first. This is possible thanks to the following technical result:

Lemma 3.2. *Given $h > 0$, we have*

$$\begin{array}{rclrcll} \mathbf{w}_h(\mathbf{x}) &=& \mathbf{e}_3, & \mathbf{a}_h(\mathbf{x}) &=& (\mathbf{e}_3 \times \mathbf{x})/2, & \forall\,\mathbf{x} \in \partial\mathcal{B}_h, \\ \mathbf{w}_h(\mathbf{x}) &=& 0, & \mathbf{a}_h(\mathbf{x}) &=& 0, & \forall\,\mathbf{x} \in \mathcal{P}. \end{array}$$

Proof. We set $\lambda = z/\delta_h(r)$ and differentiations of λ by subscripts. We have

$$\partial_z\phi_h^d(r,z) = r\lambda_z\chi_o'(\lambda), \quad \partial_r\phi_h^d(r,z) = \chi_o(\lambda) + r\lambda_r\chi_o'(\lambda). \tag{3.6}$$

Computing with the value of λ, it yields:

$$\lambda_z = \frac{1}{\delta_h(r)} \quad \lambda_r = -\frac{z\delta_h'(r)}{(\delta_h(r))^2}.$$

Our choice for χ_o implies that :

$$\chi_o(0) = \chi_o'(0) = 0, \quad \chi_o(1) = \frac{1}{2}, \quad \chi_o'(1) = 0.$$

Replacing in (3.6), λ by 0 :

$$\phi_h^d(r,z) = \partial_z\phi_h^d(r,z) = \partial_r\phi_h^d(r,z) = 0, \quad \text{on } \mathcal{P}.$$

Consequently, $\mathbf{a}_h^d = \mathbf{w}_h^d = 0$ on $\mathcal{P}$. Replacing λ by 1 :

$$\partial_z \phi_h^d(r,z) = 0, \quad \phi_h^d(r,z) = \frac{r}{2}, \quad \partial_r \phi_h^d(r,z) = \frac{1}{2} \quad \text{on } \partial\mathcal{B}_h.$$

Consequently, $\mathbf{a}_h^d(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2$ and $\mathbf{w}_h^d = \mathbf{e}_3$ on $\mathcal{B}_h$.

Concerning the smooth part, a straightforward computation leads to

$$\mathbf{w}_h^s(\mathbf{x}) = \eta_{h_0}(|\mathbf{x}-\mathbf{G}_h|-1)\mathbf{e}_3 + \eta_{h_0}'(|\mathbf{x}-\mathbf{G}_h|-1)\frac{\mathbf{x}-\mathbf{G}_h}{|\mathbf{x}-\mathbf{G}_h|} \times \frac{(\mathbf{e}_3 \times (\mathbf{x}-\mathbf{G}_h))}{2}.$$

Due to our choice, we have :

$$\eta_{h_0}(|\mathbf{x}-\mathbf{G}_h|-1) = 1, \quad \eta_{h_0}'(|\mathbf{x}-\mathbf{G}_h|-1) = 0 \quad \text{on } \partial\mathcal{B}_h.$$

Consequently $\mathbf{a}_h^s(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2$ and $\mathbf{w}_h^s = \mathbf{e}_3$ on $\partial\mathcal{B}_h$. Then,

$$\eta_{h_0}(|\mathbf{x}-\mathbf{G}_h|-1) = 0, \quad \eta_{h_0}'(|\mathbf{x}-\mathbf{G}_h|-1) = 0 \quad \text{if } |\mathbf{x}-\mathbf{G}_h| \geqslant 1+2h_0 = \sqrt{17/16}.$$

Moreover, if $\mathbf{x} \in \mathcal{P} \setminus \overline{\Omega_{h,1/4}}$, we have $r > 1/4$ and, as $\mathbf{G}_h = (0,0,1+h)$:

$$|\mathbf{x}-\mathbf{G}_h|^2 > (1+h)^2 + (1/4)^2 > 17/16.$$

Consequently, $\mathbf{a}_h^s(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathcal{P} \setminus \overline{\Omega_{h,1/4}}$.

It remains to check boundary conditions are satisfied in the transition region, *i.e.,* when $\mathbf{x} \in \overline{\Omega_{h,1/2}} \setminus \Omega_{h,1/4}$. On $\mathcal{P}$, we remark that $\mathbf{w}_h^d(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = 0 = \mathbf{a}_h^s(\mathbf{x}) = \mathbf{a}_h^d(\mathbf{x}) = 0$. Interpolating the potential vector-fields, we obtain $\mathbf{w}_h = 0$ on $\mathcal{P}$. Finally, on $\overline{\mathcal{B}}_h \cap \overline{\Omega_{h,1/4}}$ we have already computed

$$\mathbf{w}_h^d(\mathbf{x}) = \mathbf{w}_h^s(\mathbf{x}) = \mathbf{e}_3, \text{ and } \mathbf{a}_h^s(\mathbf{x}) = \mathbf{a}_h^d(\mathbf{x}) = (\mathbf{e}_3 \times \mathbf{x})/2.$$

Interpolating the potential vector-fields, we deduce $\mathbf{w}_h(\mathbf{x}) = \mathbf{e}_3$. This concludes the proof. □

REMARK 3.1. *According to this lemma, we extend* $\mathbf{a}_h$ *(respectively* $\mathbf{w}_h$ *) to* $\mathbb{R}^3_+$ *with the value* $(\mathbf{e}_3 \times \mathbf{x})/2$ *(resp.* $\mathbf{e}_3$*) in* $\mathcal{B}_h$*. In the sequel, we consider the functions* $\mathbf{a} : (h,\mathbf{x}) \to \mathbf{a}_h(\mathbf{x})$ *and* $\mathbf{w} : (h,\mathbf{x}) \to \mathbf{w}_h(\mathbf{x})$*. Denoting by* $\mathcal{Q}_c = \{(h,\mathbf{x}) \in (0,1)\times\mathbb{R}^3 \; ; \; \mathbf{x} \in \mathcal{B}_h\}$*, standard analytic arguments imply* $\mathbf{a} \in \mathcal{C}^\infty(\mathcal{Q}_c) \cap \mathcal{C}^\infty(((0,1)\times\mathbb{R}^3_+) \setminus \overline{\mathcal{Q}}_c)$*. We note that* $\mathbf{w}_h$ *vanishes as soon as* $|\mathbf{x}-\mathbf{G}_h| > (\sqrt{17/16}-1)/2$ *and* $|\mathbf{x}| > 1/2$*. Consequently, the above lemma implies* $\mathbf{w} \in H^1((\bar{h},1)\times\mathbb{R}^3_+)$ *for any* $\bar{h} > 0$ *and, after standard composition arguments, it yields*

$$\widetilde{\mathbf{w}} \in \mathcal{C}(0,T;H^1(\mathbb{R}^3_+)) \cap H^1(0,T;L^2(\mathbb{R}^3_+)).$$

as long as $h(t) \in (\bar{h},1]$ *for all* $t \in (0,T)$*. So,* $\widetilde{\mathbf{w}}$ *is a suitable test-function for the weak formulation as long as* $h(0,T) \subset (\bar{h},1)$*.*

3.3. Estimate of remainder terms. In order to exploit the weak formulation with our test-function, we need to dominate remainder terms according to **Lemma 3.1**. We begin with estimates on Sobolev norms of $\mathbf{w}_h$.

By construction, our test-functions behave differently under the sphere (in $\Omega_{h,\delta}$) and above the sphere (in $\mathcal{F}_h \setminus \Omega_{h,\delta}$ for arbitrary fixed $\delta > 0$). Above the sphere we have:

LEMMA 3.3. *Given* $\alpha \geqslant 0$ *and* $\delta > 0$ *there exists* $C(\alpha,\delta) < \infty$ *such that:*

$$\|\mathbf{a}_h\|_{H^\alpha(\mathcal{F}_h\setminus\Omega_{h,\delta})} \leqslant C(\alpha,\delta), \quad \forall\, h \in (0,1).$$

Proof. By construction the restriction $\mathbf{a} : \mathcal{Q}_{c,\delta} \to \mathbb{R}^3$, with

$$\mathcal{Q}_{c,\delta} := \{(h, \mathbf{x}) \in [0,1] \times \overline{\mathbb{R}^3_+} \quad \text{with } \mathbf{x} \notin \Omega_{h,\delta}\},$$

is smooth and with compact support. □

Inside $\Omega_{h,1/4}$, estimates rely essentially on dominations of integrals :

$$\int_0^{\frac14} \frac{r^\alpha \mathrm{d}r}{[\delta_h(r)]^\beta}$$

We let the reader refer to **Appendix** for such computations.

LEMMA 3.4. *The family $(\mathbf{w}_h)_{0<h<1}$ is uniformly bounded in $L^2(\mathbb{R}^3_+)$.*

Proof. Because of the previous lemma, we focus on $\mathbf{w}_h^d$ inside $\Omega_{h,1/4}$.

In this region, we have:

$$(\mathbf{w}_h^d(r,\theta,z))_1 = -\partial_z \phi_h^d(r,z)\cos(\theta), \quad (\mathbf{w}_h^d(r,\theta,z))_2 = -\partial_z \phi_h^d(r,z)\sin(\theta),$$

and

$$(\mathbf{w}_h^d(r,\theta,z))_3 = \partial_r \phi_h^d(r,z) + \frac{\phi_h^d(r,z)}{r}.$$

Thus

$$|\mathbf{w}_h^d(r,\theta,z)| \leqslant |\partial_z \phi_h^d(r,z)| + |\partial_r \phi_h^d(r,z)| + \frac{|\phi_h^d(r,z)|}{r}$$

Applying, **Lemma A.3**, this leads to

$$|\mathbf{w}_h^d(r,\theta,z)| \leqslant C\left(1 + \frac{r}{\delta_h(r)}\right) \quad \forall\,(r,\theta,z) \in \Omega_{h,1/4}, \quad \forall\, h \in (0,1).$$

The result then follows from **Lemma A.1** with $(\alpha,\beta) = (3,1)$. □

As technical device for applying **Lemma 3.1**, we have:

LEMMA 3.5. *Let us define*

$$w_h(r,\theta,z) = |\partial_r \mathbf{w}_h^d(r,\theta,z)| + \frac{|\partial_\theta \mathbf{w}_h^d(r,\theta,z)|}{r} + |\partial_h \mathbf{w}_h^d(r,\theta,z)|, \quad \forall\,(r,\theta,z) \in \Omega_{h,1/4}.$$

Then

$$\int_0^{\frac14} \left(\delta_h(r)^2 \left[\int_0^{\delta_h(r)} \sup_{\theta \in (0,2\pi)} |w_h(r,\theta,z)|^2 \, dz \right] \right) r\, dr \tag{3.7}$$

is uniformly bounded for $h \in (0,1)$.

Proof. A straightforward computation yields, for all $(r,\theta,z) \in \Omega_{h,1/4}$:

$$|\partial_r \mathbf{w}_h^d(r,\theta,z)| \leqslant C\left(|\partial_{rz}\phi_h^d(r,z)| + |\partial_{rr}\phi_h^d(r,z)| + \left| \frac{\partial_r \phi_h^d(r,z)}{r} - \frac{\phi_h^d(r,z)}{r^2} \right| \right)$$

and

$$\frac{|\partial_\theta \mathbf{w}_h^d(r,\theta,z)|}{r} \leqslant \frac{|\partial_z \phi_h^d(r,z)|}{r}, \quad |\partial_h \mathbf{w}_h^d(r,\theta,z)| \leqslant \frac{|\partial_h \phi_h^d(r,z)|}{r} + |\partial_{hr}\phi_h^d(r,z)| + |\partial_{hz}\phi_h^d(r,z)|.$$

Combining the above inequalities with **Lemma A.3**, we deduce there exists a constant C independent of h such that

$$|w_h(r,\theta,z)| \leqslant C\left(\frac{1}{\delta_h(r)} + \frac{r}{\delta_h(r)^2} \right)$$

for all $(r,\theta,z)\in\Omega_{h,1/4}$. Consequently,

$$\int_0^{1/4}\left(\delta_h(r)^2\left[\int_0^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}|w_h(r,\theta,z)|^2\mathrm{d}z\right]\right)r\mathrm{d}r\leqslant C\int_0^{1/4}(\delta_h(r)+1)r\mathrm{d}r.$$

As the last integral remain bounded when h goes to 0, the same holds for the integral of w_h. □

Then, to dominate the trilinear form, we need the following result:

LEMMA 3.6. *We set*

$$dw_h(r,\theta,z)=|\partial_r\mathbf{w}_h^d(r,\theta,z)|+\frac{|\partial_\theta\mathbf{w}_h^d(r,\theta,z)|}{r}+|\partial_z\mathbf{w}_h^d(r,\theta,z)|,\quad\forall\,(r,\theta,z)\in\Omega_{h,1/4}.$$

Then, the following quantity

$$\sup_{r\in(0,1/4)}\left\{\delta_h(r)^{\frac{3}{2}}\left[\int_0^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}\{|dw_h(r,\theta,z)|^2\}dz\right]^{\frac{1}{2}}\right\},\tag{3.8}$$

is uniformly bounded for $h\in(0,1)$.

Proof. As in the previous proof, there exists a constant C independent of h such that

$$|dw_h(r,\theta,z)|\leqslant C\left(\frac{1}{\delta_h(r)}+\frac{r}{\delta_h(r)^2}\right)$$

for all $(r,\theta,z)\in\Omega_{h,1/4}$. Therefore

$$\int_0^{\delta_h(r)}|dw_h(r,z)|^2\mathrm{d}z\leqslant C\left(\frac{1}{\delta_h(r)}+\frac{r^2}{\delta_h(r)^3}\right)$$

and

$$\sup_{r\in(0,1/4)}\left\{\delta_h(r)^{\frac{3}{2}}\left[\int_0^{\delta_h(r)}\sup_{\theta\in(0,2\pi)}\{|dw_h(r,\theta,z)|^2\}\mathrm{d}z\right]^{\frac{1}{2}}\right\}\leqslant C\sup_{r\in(0,1/4)}(\delta_h(r)+r),$$

which is uniformly bounded when $h\in(0,1)$. □

Finally, there holds the following lemma, which is reminiscent of works by V. Starovoitov:

LEMMA 3.7. *There exists a constant $C>0$ such that:*

$$|\nabla\mathbf{w}_h|_2^2\geqslant\frac{C}{h},\quad\forall\,h\in(0,1).$$

Proof. Given $h>0$ we already noticed that:

$$\mathbf{w}_h(\mathbf{x})=\mathbf{w}_h^d(\mathbf{x})\quad\forall\,\mathbf{x}\in\Omega_{h,1/4}.$$

Consequently:

$$|\nabla\mathbf{w}_h(\mathbf{x})|\geqslant|\partial_z\mathbf{w}_h^d(\mathbf{x})|\quad\forall\,\mathbf{x}\in\Omega_{h,1/4}.$$

With the explicit formula for $\mathbf{w}_h^d$, we have : $|\partial_z\mathbf{w}_h^d|\geqslant|\partial_{zz}\phi_h^d|$ where $|\partial_{zz}\phi_h^d(r,z)|=\dfrac{r}{\delta_h^2(r)}\chi_o''(\lambda)$. Consequently:

$$|\nabla\mathbf{w}_h|_2^2\geqslant 2\pi\int_0^{\frac{1}{4}}\frac{r^3\mathrm{d}r}{\delta_h(r)^3}\int_0^1|\chi_o''(s)|^2\mathrm{d}s.$$

As χ is a polynomial with degree 3, its second derivative does not vanish and neither does the s-integral. Then, the result yields applying **Lemma A.1** with $\alpha=\beta=3$. □

3.4. Our test function and the Stokes problem. First, we prove our choice is a good one because it is a good approximation of the solution to the Stokes problem:

LEMMA 3.8. *There exists $q_h \in \mathcal{C}_c^\infty(\overline{\mathcal{F}_h})$ such that:*

$$\left\{ \begin{array}{l} \mu\Delta \mathbf{w}_h - \nabla q_h = \mathbf{f}_h \\ \operatorname{div} \mathbf{w}_h = 0 \end{array} \right. \tag{3.9}$$

where there exists an absolute constant C for which:

$$\int_0^{2\pi}\int_0^{\frac14}\left(\delta_h(r)^2\left[\int_0^{\delta_h(r)}|\mathbf{f}_h|^2\,dz\right]\right) r\,dr\ \ d\theta + \|\mathbf{f}_h\|^2_{L^2\left(\mathcal{F}_h\setminus\Omega_{h,1/4}\right)} \leqslant C$$

Proof. By construction, we have $\mathbf{w}_h = \nabla\times\widetilde{\mathbf{a}}_h^d + \nabla\times\widetilde{\mathbf{a}}_h^s$ where

$$\widetilde{\mathbf{a}}_h^d(\mathbf{x}) = \left\{ \begin{array}{rl} \eta_{1/2}(r)\mathbf{a}_h^d(\mathbf{x}), & \mathbf{x}\in\Omega_{h,1/2}, \\ 0 & \text{else} \end{array}\right. \qquad \widetilde{\mathbf{a}}_h^s(\mathbf{x}) = \left\{ \begin{array}{rl} (1-\eta_{1/2}(r))\mathbf{a}_h^s(\mathbf{x}), & \mathbf{x}\in\Omega_{h,1/2}, \\ \mathbf{a}_h^s(\mathbf{x}), & \text{else.} \end{array}\right.$$

Then, according to **Lemma 3.3**, the smooth part $\widetilde{\mathbf{a}}_h^s$ is bounded in any Sobolev space uniformly in h. Consequently, $\widetilde{\mathbf{f}}_h = \mu\Delta\nabla\times\widetilde{\mathbf{a}}_h^s$ is bounded in all Sobolev spaces. We have:

$$\mu\Delta\mathbf{w}_h = \mu\Delta\nabla\times\widetilde{\mathbf{a}}_h^d + \widetilde{\mathbf{f}}_h.$$

In the following we write $\widetilde{\phi}_h^d(r,z) = \eta_{1/2}(r)\phi_h^d(r,z)$ for all $(r,\theta,z)\in\Omega_{h,1/2}$. Let us recall that in cylindrical coordinates, we have:

$$\Delta = \frac{\partial_r[r\partial_r]}{r} + \frac{\partial_{\theta\theta}}{r^2} + \partial_{zz}.$$

Consequently,

$$\left[\Delta\nabla\times\widetilde{\mathbf{a}}_h^d\right]_1 = -\left[\partial_{rrz}\widetilde{\phi}_h^d + \frac{\partial_{rz}\widetilde{\phi}_h^d}{r} - \frac{\partial_z\widetilde{\phi}_h^d}{r^2} + \partial_{zzz}\widetilde{\phi}_h^d\right]\cos(\theta)$$

and

$$\left[\Delta\nabla\times\widetilde{\mathbf{a}}_h^d\right]_2 = -\left[\partial_{rrz}\widetilde{\phi}_h^d + \frac{\partial_{rz}\widetilde{\phi}_h^d}{r} - \frac{\partial_z\widetilde{\phi}_h^d}{r^2} + \partial_{zzz}\widetilde{\phi}_h^d\right]\sin(\theta)$$

$$\left[\Delta\nabla\times\widetilde{\mathbf{a}}_h^d\right]_3 = \partial_{rrr}\widetilde{\phi}_h^d + 2\frac{\partial_{rr}\widetilde{\phi}_h^d}{r} - \frac{\partial_r\widetilde{\phi}_h^d}{r^2} + \frac{\widetilde{\phi}_h^d}{r^3} + \partial_{zz}\left[\partial_r\widetilde{\phi}_h^d + \frac{\widetilde{\phi}_h^d}{r}\right].$$

We remark here that

$$\partial_{zzz}\widetilde{\phi}_h^d(r,z) = -6\frac{r\,\eta_{1/2}(r)}{\delta_h^3(r)}.$$

Consequently, denoting by Φ a primitive of $s\mapsto -6s\eta_{1/2}(s)/(\delta_h(s))^3$, we have

$$\nabla\Phi(r) = (\partial_{zzz}\phi_h^d\cos(\theta), \partial_{zzz}\phi_h^d\sin(\theta), 0)^\top.$$

We set:

$$q_h(\mathbf{x}) = \mu\Phi(r) + \mu\partial_z\left[\partial_r\widetilde{\phi}_h^d + \frac{\widetilde{\phi}_h^d}{r}\right], \quad \check{\mathbf{f}}_h = \mu\Delta\nabla\times\widetilde{\mathbf{a}}_h^d - \nabla q_h.$$

In particular $\mu\Delta\mathbf{w}_h - \nabla q_h = \widetilde{\mathbf{f}}_h + \check{\mathbf{f}}_h$, so that our result follows from the same result for $\check{\mathbf{f}}_h$. Denoting by $\check{f}_1, \check{f}_2, \check{f}_3$ the Cartesian components of $\check{\mathbf{f}}_h$, straightforward computations show that:

$$|\check{f}_1|^2 + |\check{f}_2|^2 \leqslant 4\left[\partial_{rrz}\widetilde{\phi}^d_h + \frac{\partial_{rz}\widetilde{\phi}^d_h}{r} - \frac{\partial_z\widetilde{\phi}^d_h}{r^2}\right]^2.$$

As $\eta'_{1/2}$ vanishes uniformly in $\Omega_{h,1/4}$, **Lemma 3.3** implies there exists a universal constant C such that:

$$|\check{f}_1|^2 + |\check{f}_2|^2 \leqslant C\left[1 + \left|\partial_{rrz}\phi^d_h + \frac{\partial_{rz}\phi^d_h}{r} - \frac{\partial_z\phi^d_h}{r^2}\right|\right]^2.$$

Then, for the same reasons, we have:

$$|\check{f}_3|^2 \leqslant C\left[1 + \left|\partial_{rrr}\phi^d_h\right| + \left|\frac{2\partial_{rr}\phi^d_h}{r}\right| + \left|\frac{\phi^d_h}{r^3} - \frac{\partial_r\phi^d_h}{r^2}\right|\right]^2.$$

Replacing with the size computed in **Lemma A.3**, we obtain:

$$|\check{\mathbf{f}}_h(\mathbf{x})|^2 \leqslant C\left(\frac{r}{\delta_h^2(r)} + \frac{1}{\delta_h(r)}\right)^2,$$

Consequently, for arbitrary $r \in (0, 1/2)$

$$\int_0^{\delta_h(r)} |\check{\mathbf{f}}_h|^2 \mathrm{d}z \leqslant C\left(\frac{r^2}{\delta_h(r)^3} + \frac{1}{\delta_h(r)}\right)$$

And:

$$\int_0^{2\pi}\int_0^{1/2} \delta_h(r)^2 \int_0^{\delta_h(r)} |\check{\mathbf{f}}_h|^2 \mathrm{d}z r \mathrm{d}r \mathrm{d}\theta \leqslant C\int_0^{1/2}\left(\frac{r^3}{\delta_h(r)} + r\delta_h(r)\right)\mathrm{d}r$$

which is uniformly bounded for $h \in (0,1)$. This concludes the proof. □

As a direct corollary, we get:

LEMMA 3.9. *There exists $K_m < \infty$, and a function $\widetilde{n}_3 : [0,1] \to \mathbb{R}_+$ such that, for any $h < 1$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G}_h)$ such that $\mathbf{w} = \mathbf{V}_\mathbf{w} + \mathbf{R}_\mathbf{w} \times (\mathbf{x} - \mathbf{G}_h)$ in $\mathcal{B}_h$, we have:*

$$\left|2\mu\int_{\mathbb{R}^3_+} D(\mathbf{w}_h) : D(\mathbf{w})\ d\mathbf{x} - \widetilde{n}_3(h)\mathbf{V}_\mathbf{w}\cdot\mathbf{e}_3\right| \leqslant K_m\|\nabla\mathbf{w}\|_{L^2(\mathbb{R}^3_+)}, \tag{3.10}$$

Moreover, there exist $h_m > 0$ and a constant $c > 0$ such that $\widetilde{n}_3(h) \geqslant c/h$, for all $h < h_m$.

Proof. Given $h > 0$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G}_h)$, we apply the Stokes identity with (3.9) and we obtain:

$$2\mu\int_{\mathbb{R}^3_+} \mathrm{D}(\mathbf{w}_h) : \mathrm{D}(\mathbf{w})\ \mathrm{d}\mathbf{x} = \int_{\partial\mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n}\cdot\mathbf{w}\ \mathrm{d}\sigma - \int_{\mathcal{F}_h} \mathbf{f}_h\cdot\mathbf{w}\ \mathrm{d}\mathbf{x}. \tag{3.11}$$

For symmetry reasons, there exists $\widetilde{n}_3 : (0,1) \to \mathbb{R}$ such that:

$$\int_{\partial\mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n}\ \mathrm{d}\sigma = \widetilde{n}_3(h)\mathbf{e}_3,$$

and

$$\int_{\partial\mathcal{B}_h} (\mathbf{x} - \mathbf{G}_h) \times \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n}\ \mathrm{d}\sigma = 0.$$

On the other hand, applying (3.2) and **Lemma 3.8**, we also deduce

$$\left|\int_{\mathcal{F}_h} \mathbf{f}_h \cdot \mathbf{w}\ \mathrm{d}\mathbf{x}\right| \leqslant C\|\nabla \mathbf{w}\|_{L^2(\mathbb{R}^3_+)}, \quad \forall\, h \in (0,1),$$

where C is a positive constant. Finally, we have obtained the existence of a constant K such that, for arbitrary $h \in (0,1)$ and $\mathbf{w} \in \mathbb{V}(\mathbf{G})$,

$$\left|2\mu \int_{\mathbb{R}^3_+} \mathrm{D}(\mathbf{w}_h) : \mathrm{D}(\mathbf{w})\ \mathrm{d}\mathbf{x} - \widetilde{n}_3(h)\mathbf{V}_{\mathbf{w}} \cdot \mathbf{e}_3\right| \leqslant K\|\nabla \mathbf{w}\|_{L^2(\mathbb{R}^3_+)}.$$

In order to estimate $\widetilde{n}_3$, we take $\mathbf{w} = \mathbf{w}_h$ in (3.11). and we obtain

$$\widetilde{n}_3 = \int_{\partial\mathcal{B}_h} \mathbb{T}(\mathbf{w}_h, p_h)\mathbf{n} \cdot \mathbf{e}_3\ \mathrm{d}\sigma = 2\mu \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2\ \mathrm{d}\mathbf{x} + \int_{\mathcal{F}_h} \mathbf{f}_h \cdot \mathbf{w}_h\ \mathrm{d}\mathbf{x}. \tag{3.12}$$

Dealing as previously with the last integral, we deduce that:

$$\left|\int_{\mathcal{F}_h} \mathbf{f}_h \cdot \mathbf{w}_h\ \mathrm{d}\mathbf{x}\right| \leqslant K\|\nabla \mathbf{w}_h\|_{L^2(\mathbb{R}^3_+)}, \quad \forall\, h \in (0,1).$$

But, applying **Lemma 3.7** we have that

$$2\mu \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2\ \mathrm{d}\mathbf{x} = \mu \int_{\mathbb{R}^3_+} |\nabla \mathbf{w}_h|^2\ \mathrm{d}\mathbf{x} \geqslant \frac{C}{h} \quad \forall\, h \in (0,1).$$

Consequently, the asymptotic behavior of the right-hand side in (3.12) when h goes to 0 is prescribed by the first integral. Hence, there exists $h_m > 0$ and constants $\widetilde{c}, c > 0$ such that :

$$\widetilde{n}_3(h) \geqslant \widetilde{c} \int_{\mathbb{R}^3_+} |D(\mathbf{w}_h)|^2\ \mathrm{d}\mathbf{x} \geqslant \frac{c}{h}, \quad \forall\, h \in (0, h_m).$$

□

4. Proof of Theorem 1.1. We let the reader convince himself that Theorem 1.1 is a direct consequence to

THEOREM 4.1. *Given* $(\mathbf{u}, \mathbf{G})$ *a weak solution to (FSIS) on* $(0,T)$ *with initial data* $(\mathbf{u}^0, \mathbf{G}^0)$, *we assume there exists* $0 \leqslant \tau_0 < \tau_1 \leqslant T$ *for which*

$$h(t) := \mathrm{dist}(\mathcal{B}(t), \mathcal{P}) \leqslant 1 \quad \forall\, t \in [\tau_0, \tau_1].$$

Then, there exists $C(\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)}) < \infty$ *depending only on the* L^2*-norm of initial data such that,*

$$h(t) \geqslant h(\tau_0) \exp\left[-C(\|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)})(1 + \sqrt{T})\right], \quad \forall\, t \in (\tau_0, \tau_1).$$

The remainder of this paper is devoted to the proof of this result. From now on $(\mathbf{u}, \mathbf{G})$ is a given weak solution to (FSIS) with initial data $(\mathbf{u}^0, \mathbf{G}^0)$. For simplicity, we assume that $h(t) \leqslant 1$ for all $t \in (0,T)$. This means that $\tau_0 = 0$ and $\tau_1 = T$ in the assumptions of our theorem.

As mentioned before, we estimate the distance h from below with our approximation of the Stokes problem. So, from now on, $(\mathbf{w}_h)_{h \in (0,1)}$ are the approximations constructed in Section 3.1. Given $0 < t_0 < t_1 < 1$, we set :

$$\zeta_\varepsilon(t) = \eta_\varepsilon(\mathrm{dist}(t, [t_0, t_1])).$$

Then, $\zeta_\varepsilon \in \mathcal{D}(0,T)$ whenever ε is sufficiently small. Consequently, according to Remark 3.1, for ε sufficiently small

$$\begin{array}{rccl} \widetilde{\mathbf{w}}_\varepsilon : & (0,T) \times \mathbb{R}^3_+ & \longrightarrow & \mathbb{R}^3, \\ & (t, \mathbf{x}) & \longmapsto & \zeta_\varepsilon(t)\mathbf{w}_{h(t)}(x_1 - G_1(t), x_2 - G_2(t), x_3) \end{array}$$

can be taken as test function in (2.3):

$$\int_{(0,T)\times\mathbb{R}^3_+} [\rho_{\mathbf{G}}\mathbf{u}\cdot\partial_t\widetilde{\mathbf{w}}_\varepsilon + (\mathbf{u}\otimes\mathbf{u} - 2\mu\mathrm{D}(\mathbf{u})) : \mathrm{D}(\widetilde{\mathbf{w}}_\varepsilon)]\ d\mathbf{x}\ dt = 0 \tag{4.1}$$

In the following, we set:

$$\begin{aligned} I_1 &:= \int_{(0,T)\times\mathbb{R}^3_+} \rho_{\mathbf{G}}\mathbf{u}\cdot\partial_t\widetilde{\mathbf{w}}_\varepsilon\ d\mathbf{x}, \\ I_2 &:= \int_{(0,T)\times\mathbb{R}^3_+} \mathbf{u}\otimes\mathbf{u} : D(\widetilde{\mathbf{w}}_\varepsilon)\ d\mathbf{x} \\ I_3 &:= \int_{(0,T)\times\mathbb{R}^3_+} D(\mathbf{u}) : D(\widetilde{\mathbf{w}}_\varepsilon)\ d\mathbf{x}. \end{aligned}$$

After a change of variables, we have for almost all $t\in(0,T)$:

$$\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ d\mathbf{x} = \zeta_\varepsilon(t)\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})\ d\mathbf{x}.$$

Thus, applying Lemma 3.9:

$$\int_{\mathbb{R}^3_+} D(\mathbf{u})(t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})\ d\mathbf{x} = \dot{h}\widetilde{n}_3(h) + E(t)$$

where $|E(t)| = K_M|\nabla\mathbf{u}(t,\cdot)|_2$. Consequently:

$$I_3 = \int_0^T \zeta_\varepsilon(t)\dot{h}(t)\widetilde{n}_3(h(t))\mathrm{d}t + \widetilde{E} \tag{4.2}$$

where

$$|\widetilde{E}| \leqslant K_M\sqrt{T}\|\mathbf{u}_0\|_2. \tag{4.3}$$

Similarly, for almost all $t\in(0,T)$:

$$\begin{aligned} &\int_{\mathbb{R}^3_+} [\mathbf{u}\otimes\mathbf{u}](t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ \mathrm{d}\mathbf{x} \\ &\qquad = \zeta_\varepsilon(t)\int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}_h}\mathbf{u}\otimes\mathbf{u}](t,x_1+G_1,x_2+G_2,x_3) : D(\mathbf{w}_{h(t)})(\mathbf{x})\ \mathrm{d}\mathbf{x}. \end{aligned}$$

Consequently, applying Lemma 3.1 together with Lemmata 3.6 and 3.3, we obtain:

$$\left|\int_{\mathbb{R}^3_+} [\mathbf{u}\otimes\mathbf{u}](t,\cdot) : D(\widetilde{\mathbf{w}}_\varepsilon)(t,\cdot)\ \mathrm{d}\mathbf{x}\right| \leqslant K_m\|\nabla\mathbf{u}\|^2_{L^2(\mathbb{R}^3_+)}.$$

Thus,

$$|I_2| \leqslant K_m\|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)}. \tag{4.4}$$

Finally, computing $\partial_t\widetilde{\mathbf{w}}_\varepsilon$ we have, after our change of variable,

$$I_1 = I_1^\chi + I_1^w$$

where:

$$I_1^{\chi} := \int_0^T \int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}_h} \mathbf{u}](t, x_1 + G_1, x_2 + G_2, x_3)] \cdot \zeta_\varepsilon'(t) \mathbf{w}_{h(t)}(\mathbf{x}) \, \mathrm{d}\mathbf{x}$$

and

$$I_1^w := \int_0^T \zeta_\varepsilon(t) \int_{\mathbb{R}^3_+} [\rho_{\mathbf{G}_h} \mathbf{u}](x_1 + G_1, x_2 + G_2, x_3) \cdot \left[\dot{h} \partial_h \mathbf{w}_h - V_1 \partial_{x_1} \mathbf{w}_h - V_2 \partial_{x_2} \mathbf{w}_h\right] (\mathbf{x}) \, \mathrm{d}\mathbf{x}.$$

Applying the Cauchy–Schwarz inequality and Lemma 3.4 on $\mathbf{w}_h$, we deduce that:

$$|I_1^{\chi}| \leqslant C \left[\int_{t_0 - \varepsilon}^{t_0} |\zeta_\varepsilon'(t)| \ \|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3_+)} \, \mathrm{d}t + \int_{t_1}^{t_1 + \varepsilon} |\zeta_\varepsilon'(t)| \ \|\mathbf{u}(t, \cdot)\|_{L^2(\mathbb{R}^3_+)} \, \mathrm{d}t \right],$$

and therefore, using the uniform L^2-bound on $\mathbf{u}$ we obtain

$$|I_1^{\chi}| \leqslant K \|\mathbf{u}^0\|_{L^2(\mathbb{R}^3_+)}. \tag{4.5}$$

Finally, applying (3.2) in Lemma 3.1 together with (3.7) in Lemma 3.5, we conclude that:

$$|I_1^w| \leqslant K_m \int_0^T [|\dot{h}|^2 + |\mathbf{V}_1|^2 + |\mathbf{V}_2|^2]^{\frac{1}{2}} \|\nabla \mathbf{u}\|_{L^2(\mathbb{R}^3_+)} \ dt.$$

so that, with standard energy estimate:

$$|I_1^w| \leqslant K_m \sqrt{T} \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \tag{4.6}$$

Gathering (4.2), (4.4), (4.5), (4.6) with (4.1) yields

$$\left| \int_0^T \zeta_\varepsilon(t) \dot{h}(t) \widetilde{n}_3(h(t)) \ dt \right| \leqslant K_m (1 + \sqrt{T}) \left\{ \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \right\},$$

where we emphasize that K_m depend only on our choice for the approximation of the solution to the Stokes problem, but not on ε. Thus, letting ε go to 0, as h and $\widetilde{n}_3$ are continuous function, we obtain:

$$|N_3(h(t_1)) - N_3(h(t_0))| \leqslant K_m (1 + \sqrt{T}) \left\{ \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \right\},$$

where N_3 is a primitive of $\widetilde{n}_3$ which vanishes in $h = 1$ for example. Applying Lemma 3.9, we have $\widetilde{n}_3(h) \geqslant c/h$ when $0 < h < h_m$ for some $c > 0$ and $h_m > 0$ and we finally deduce:

$$|\ln(h(t)/h(t_0))| \leqslant K_m (1 + \sqrt{T}) \left\{ \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \right\}.$$

Because h is continuous, letting t_0 tend to 0, we finally obtain:

$$h(t) \geqslant h(0) \exp\left[-K_m (1 + \sqrt{T}) \left\{ \|\mathbf{u}_0\|_{L^2(\mathbb{R}^3_+)} + \|\mathbf{u}_0\|^2_{L^2(\mathbb{R}^3_+)} \right\} \right].$$

This is the expected result.

Appendix A. Detailed description of ϕ_h^d. In this section we estimate the size of ϕ_h^d and its derivatives. We recall that :

$$\phi_h^d(r, \theta, z) = r \chi_o(z/\delta_h(r)), \quad \text{with} \quad \chi_o(s) = \frac{s^2(3 - 2s)}{2}.$$

In order to compare functions in the following, we introduce the following conventions. Given families $(f_h : \Omega_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$ and $(g_h : \Omega_{h,1/4} \to \mathbb{R})_{h\in(0,1)}$ we denote by $f_h \prec g_h$ if there exists an absolute constant such that:

$$|f_h(\mathbf{x})| \leqslant C g_h(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega_{h,1/4} \text{ and } h < 1.$$

Given non negative functions $f : (0,1) \to \mathbb{R}^+$ and $g : (0,1) \to \mathbb{R}^+$, we also denote by

$$f(s) \sim g(s) \quad \forall s \in (0,1),$$

if there exist two positive constants c and C such that

$$cf(s) \leqslant g(s) \leqslant Cf(s) \quad \forall s \in (0,1).$$

First, we compute typical $L^1(0,1/4)$-sizes of functions $r \mapsto r^\alpha/(\delta_h(r))^\beta$

LEMMA A.1. *Given $(\alpha,\beta) \in (\mathbb{R}_+)^2$, we have the following estimations for all $h \in (0,1)$:*

$$\int_0^{\frac{1}{4}} \frac{r^\alpha}{\delta_h(r)^\beta} dr \sim \begin{cases} 1 & \text{if } \alpha > 2\beta - 1, \\ h^{\frac{(\alpha+1)-2\beta}{2}} & \text{if } \alpha < 2\beta - 1, \end{cases}$$

Proof. As in [8], we remark that, for all $h \in (0,1)$, we have:

$$h + \frac{s^2}{2} \leqslant \delta_h(s) \leqslant h + s^2 \quad \forall s \in (0,1/4).$$

Consequently, we can replace $\delta_h(r)$ by $h + \gamma r^2$ with some generic parameter $\gamma > 0$ and we are bound to calculate:

$$I_{\alpha,\beta} := \int_0^{1/4} \frac{r^\alpha}{(h+\gamma r^2)^\beta} \mathrm{d}r$$

in which we set $r = \sqrt{h}s$. It yields:

$$I_{\alpha,\beta} := h^{\frac{(\alpha+1)-2\beta}{2}} \int_0^{\frac{1}{4\sqrt{h}}} \frac{s^\alpha}{(1+\gamma s^2)^\beta} \mathrm{d}s$$

Consequently, if $\alpha > 2\beta - 1$, the integral behaves like $Ch^{-\frac{(\alpha+1)-2\beta}{2}}$ and we obtain the first case. While if $\alpha < 2\beta - 1$ the integral goes to a finite positive value as $h \to \infty$ and we obtain the second case. $\square$

We now compare $\lambda(r,z,h) = z/\delta_h(r)$ to members functions $(r,\theta,z) \mapsto r^\alpha/(\delta_h(r))^\beta$ in $\Omega_{h,1/4}$.

LEMMA A.2. *We have the following sizes*

$$\begin{array}{llll} \lambda \prec 1, & \lambda_r \prec r/\delta_h, & \lambda_z \prec 1/\delta_h, & \lambda_h \prec 1/\delta_h, \\ \lambda_{rh} \prec r/\delta_h^2, & \lambda_{zh} \prec r/\delta_h^2, & \lambda_{rr} \prec 1/\delta_h, & \lambda_{rz} \prec r/\delta_h^2, \\ \lambda_{rrz} \prec 1/\delta_h^2, & & \lambda_{rrr} \prec r/\delta_h^2. & \end{array}$$

Proof. The reader may rapidly check that all the derivatives of δ_h are independent of h and that, all the odd ones are bounded by r over $(0,1/4)$. Then, in $\Omega_{h,1/4}$ we have $z \in (0,\delta_h(r))$, consequently $\lambda \prec 1$. Then

$$\lambda_r = -\frac{\lambda\delta_h'}{\delta_h}, \quad \lambda_z = \frac{1}{\delta_h}, \quad \lambda_h = -\frac{\lambda}{\delta_h}.$$

As δ' is bounded by r necessarily independent of h, we get $\lambda_r \prec r/\delta_h$ and $\lambda_z \prec 1/\delta_h$, $\lambda_h \prec 1/\delta_h$. To the next order, we get:

$$\lambda_{rz} = -\frac{\delta_h'}{\delta_h^2} \quad \lambda_{rr} = \lambda\left(2\frac{(\delta_h')^2}{\delta_h^2} - \frac{\delta_h''}{\delta_h}\right), \quad \lambda_{rh} = -\frac{\lambda\delta_h'}{\delta_h^2}, \quad \lambda_{zh} = -\frac{1}{\delta_h^2}.$$

As δ'' is bounded independently of h and $r^2 \leqslant h + r^2$, we obtain:

$$\lambda_{rz} \prec \frac{r}{\delta_h^2}, \quad \lambda_{rr} \prec \frac{1}{\delta_h}, \quad \lambda_{rh} \prec \frac{r}{\delta_h}, \quad \lambda_{zh} \prec \frac{1}{\delta_h^2}.$$

Finally,

$$\lambda_{rrz} = \frac{1}{\delta_h}\left(2\frac{(\delta_h')^2}{\delta_h^2} - \frac{\delta_h''}{\delta_h}\right), \quad \lambda_{rrr} = \lambda\left(6\frac{\delta_h''\delta_h'}{\delta_h^2} - 6\frac{(\delta_h')^3}{\delta_h^3} - \frac{\delta_h^{(3)}}{\delta_h}\right)$$

And, as $\delta_h^{(3)}$ is bounded by r and $r^2 \leqslant \delta_h$,

$$\lambda_{rrz} \prec \frac{1}{\delta_h^2}, \quad \lambda_{rrr} \prec \frac{r}{\delta_h^2}.$$

□

Then, we obtain:

Lemma A.3. *We have the following sizes:*

$$\begin{array}{llll}
\phi_h^d \prec r, & \partial_r\phi_h^d \prec 1, & \partial_z\phi_h^d \prec r/\delta_h, & \partial_r\phi_h^d/r - \phi_h^d/r^2 \prec r/\delta_h \\
\partial_h\phi_h^d \prec r/\delta_h, & \partial_{rh}\phi_h^d \prec 1/\delta_h, & \partial_{zh}\phi_h^d \prec r/\delta_h^2, & \partial_{rz}\phi_h^d/r - \partial_z\phi_h^d/r^2 \prec r/\delta_h^2 \\
\partial_{rr}\phi_h^d \prec r/\delta_h, & \partial_{rz}\phi_h^d \prec 1/\delta_h, & \partial_{zz}\phi_h^d \prec r/\delta_h^2, & \\
\partial_{rrr}\phi_h^d \prec 1/\delta_h, & \partial_{rzz}\phi_h^d \prec 1/\delta_h^2, & \partial_{rrz}\phi_h^d \prec r/\delta_h^2, & \partial_{zzz}\phi_h^d \prec r/\delta_h^3.
\end{array}$$

Proof. By definition, we have $\phi_h^d(r,z) = r\chi_o(\lambda)$, where χ_o is a fixed polynomial, and, according to the previous lemma, λ is bounded. Consequently, we obtain $\phi_h^d \prec r$.

In the following, we shall drop all dependencies of χ in λ. Due to the same argument as for χ_o, all these quantities depending only on χ_o are bounded independently of (h, r, z) in $\Omega_{h,1/4}$.

So, we compute:

$$\partial_r\phi_h^d = \chi_o + r\lambda_r\chi_o', \quad \partial_z\phi_h^d = r\lambda_z\chi_o', \quad \partial_h\phi_h^d = r\lambda_h\chi_o'.$$

Applying the previous lemma and $r^2 \leqslant \delta_h(r)$ we get:

$$\partial_r\phi_h^d \prec 1, \quad \partial_z\phi_h^d \prec r/\delta_h, \quad \partial_h\phi_h^d \prec r/\delta_h, \quad \partial_r\phi_h^d/r - \phi_h^d/r^2 = \lambda_r\chi_o' \prec r/\delta_h.$$

To the next order, we obtain, as λ_z is independent of z :

$$\partial_{rr}\phi_h^d = (2\lambda_r + r\lambda_{rr})\chi_o' + r(\lambda_r)^2\chi_o'', \quad \partial_{rz}\phi_h^d = (\lambda_z + r\lambda_{rz})\chi_o' + r\lambda_r\lambda_z\chi_o'' \quad \partial_{zz}\phi_h^d = r(\lambda_z)^2\chi_o''.$$

$$\partial_{zh}\phi_h^d = r\lambda_z\lambda_h\chi_o'' + r\lambda_{hz}\chi_o' \quad \text{and} \quad \partial_{rh}\phi_h^d = \lambda_h\chi_o' + r\lambda_{rh}\chi_o' + r\lambda_r\lambda_h\chi_o''.$$

As previously :

$$\partial_{rr}\phi_h^d \prec r/\delta_h, \quad \partial_{rz}\phi_h^d \prec 1/\delta_h \quad \partial_{zz}\phi_h^d \prec r/\delta_h^2 \quad \partial_{rh}\phi_h^d \prec 1/\delta_h, \quad \partial_{zh}\phi_h^d \prec r/\delta_h^2.$$

and

$$\frac{\partial_{rz}\phi_h^d}{r} - \frac{\partial_z\phi_h^d}{r^2} = \lambda_{rz}\chi_o' + \lambda_r\lambda_z\chi_o'' \prec r/\delta_h^2$$

To the next order, we obtain:

$$\partial_{rrr}\phi_h^d = (3\lambda_{rr} + r\lambda_{rrr})\chi_o' + (3\lambda_r^2 + 3r\lambda_{rr}\lambda_r))\chi_o'' + r(\lambda_r)^3\chi_o^{(3)},$$

thus $\partial_{rrr}\phi_h^d \prec 1/\delta_h$; and:

$$\partial_{rzz}\phi_h^d = (\lambda_z^2 + 2r\lambda_{rz}\lambda_z)\chi_o'' + r(\lambda_z)^2\lambda_r\chi_o^{(3)},$$

so $\partial_{rzz}\phi_h^d \prec 1/\delta_h^2$; and

$$\partial_{rrz}\phi_h^d = (2\lambda_{rz} + r\lambda_{rrz})\chi_o' + (2\lambda_r\lambda_z + r(\lambda_{rr}\lambda_z + 2\lambda_{rz}\lambda_r)\chi_o'' + r(\lambda_r)^2\lambda_z\chi_o^{(3)}.$$

so $\partial_{rrz}\phi_h^d \prec r/\delta_h^2$. Finally, $\partial_{zzz}\phi_h^d = r(\lambda_z)^3\chi_o^{(3)}$, so that $\partial_{zzz}\phi_h^d \prec r/{\delta_h}^3$. $\square$

REFERENCES

[1] M.D.A. Cooley and M.E. O'Neill, *On the slow motion generated in a viscous fluid by the approach of a sphere to a plane wall or stationary sphere.*, Mathematika, 16 (1969), pp. 37–49.

[2] Patricio Cumsille and Takéo Takahashi, *Wellposedness for the system modelling the motion of a rigid body of arbitrary form in an incompressible viscous fluid*, Czechoslovak Mathematical Journal, (to appear).

[3] Patricio Cumsille and Marius Tucsnak, *Wellposedness for the Navier-Stokes flow in the exterior of a rotating obstacle*, Math. Methods Appl. Sci., 29 (2006), pp. 595–623.

[4] Eduard Feireisl, *On the motion of rigid bodies in a viscous incompressible fluid*, J. Evol. Equ., 3 (2003), pp. 419–441. Dedicated to Philippe Bénilan.

[5] Giovanni P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, vol. 38 of Springer Tracts in Natural Philosophy, Springer-Verlag, New York, 1994. Linearized steady problems.

[6] Max D. Gunzburger, Hyung-Chun Lee, and Gregory A. Seregin, *Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions*, J. Math. Fluid Mech., 2 (2000), pp. 219–266.

[7] T.I. Hesla, *Collisions of smooth bodies in viscous fluids : A mathematical investigation*, PhD thesis, University of Minnesota, revised version, October 2005.

[8] M. Hillairet, *Lack of collision between solid bodies in a 2D incompressible viscous flow*, Comm. Partial Differential Equations, 32 (2007), pp. 1345–1371.

[9] Atsushi Inoue and Minoru Wakimoto, *On existence of solutions of the Navier-Stokes equation in a time dependent domain*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 24 (1977), pp. 303–319.

[10] V.N. Starovoĭtov, *Nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid*, J. Math. Sci., 130 (2005), pp. 4893–4898.

[11] Victor N. Starovoitov, *Behavior of a rigid body in an incompressible viscous fluid near a boundary*, in Free boundary problems (Trento, 2002), vol. 147 of Internat. Ser. Numer. Math., Birkhäuser, Basel, 2004, pp. 313–327.

[12] Takéo Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, 8 (2003), pp. 1499–1532.

[13] R. Temam, *Problèmes mathématiques en plasticité*, Gauthier-Villars, Montrouge, 1983.

[14] Juan Luis Vázquez and Enrique Zuazua, *Lack of collision in a simplified 1D model for fluid-solid interaction*, Math. Models Methods Appl. Sci., 16 (2006), pp. 637–678.